Contents

1 The Complex Plane

1.1 Algebraic Structure and Polar Form

Definition 1.1

The complex numbers $\mathbb{C}$ is $\mathbb{R}^2$ with the following operations:

1. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
2. $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

Theorem 1.1 **Field Structure of $\mathbb{C}$**

$\mathbb{C}$ is a field with $0 = (0, 0)$, $1 = (1, 0)$, and

$$-(x, y) = (-x, -y), \quad (x, y)^{-1} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$

for every $(x, y) \neq (0, 0)$.

Notation

If $z = (x, y)$, we call x the "real part" of z and write $x = \operatorname{Re}(z)$. Call y the imaginary part of z and write $y = \operatorname{Im}(z)$.

Definition 1.2

$i = (0, 1) \in \mathbb{C}$. For $z \in \mathbb{C}$, we have $z = \operatorname{Re}(z) + \operatorname{Im}(z) \cdot i$.

Example

$i^2 = -1$ and $(-i)^2 = (0, -1)^2 = -1$.

Definition 1.3 **Conjugation**

Define conjugation to be the map $^- : \mathbb{C} \to \mathbb{C}$ given by

$$z = x + iy \mapsto x - iy.$$

Note

$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$ and $\operatorname{Im}(z) = \frac{z - \bar{z}}{2i} = \frac{2yi}{2i} = y$.

Definition 1.4 **Polar Representation**

If $z = x + iy \in \mathbb{C}$, it has a polar representation $(x, y) \rightsquigarrow (r, \theta)$ with $r = \sqrt{x^2 + y^2}$, $r \cos\theta = x$ and $r \sin\theta = y$.

Note

If z has polar coordinates (r, θ), then $z = r\cos\theta + ir\sin\theta$.

Remark

If z_1, z_2 have polar representations $z_1 : (r_1, \theta_1)$ and $z_2 : (r_2, \theta_2)$, then

$$\begin{aligned} z_1 \cdot z_2 &= (r_1\cos\theta_1 + ir_1\sin\theta_1) \cdot (r_2\cos\theta_2 + ir_2\sin\theta_2) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

1.2 Metric Geometry and Topology of $\mathbb{C}$

Note

$\mathbb{C}$ is a metric space with $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ given by $d(x, y) = |x - y|$, where $|\,| : \mathbb{C} \to [0, \infty)$ is defined to be the 2-norm, so the triangle inequality holds.

Remark

$\mathbb{C}$ is a topological space, so topological properties hold.

Definition 1.5

If z has polar representation (r, θ), we call θ the argument of z. Write $\theta = \arg(z)$. We'll think about arg as a multi-valued function. We write $\operatorname{Arg}(z) = \theta$ if $\theta \in (-\pi, \pi]$, and call it the principal value.

Example

If $z = r(\cos\theta + i\sin\theta) \neq 0$ and $n \geq 1$, then z has exactly n distinct nth roots.

2 Complex Differentiation and Holomorphic Functions

2.1 Complex Differentiability

Definition 2.1

Suppose $f : U \to \mathbb{C}$ is such that $U \subseteq \mathbb{C}$ is an open set with $z_0 \in U$. We say that f has derivative $f'(z_0)$ at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

We say that f is differentiable at z_0 if the limit above exists.

Theorem 2.1 **Product Rule**

If f and g are differentiable at z_0, then so is $(f \cdot g)$ and

$$(f \cdot g)' = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$

2.2 Cauchy-Riemann Equations and Holomorphicity

Remark

If $f : \mathbb{C} \to \mathbb{C}$, we may write $f(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$. If f is differentiable at $z_0 = x_0 + iy_0$, say $f'(z_0) = a + ib$, then if we let $z = x + iy$ and $z_0 = x_0 + iy_0$, we have

$$\begin{aligned} R(z) &= f(z) - f(z_0) - f'(z_0)(z - z_0) \\ &= (u(z) - u(z_0) - a(x - x_0) + b(y - y_0)) + \\ &\quad i(v(z) - v(z_0) - a(y - y_0) - b(x - x_0)) \\ &= R_1(z) + iR_2(z) \end{aligned}$$

Then, since f is differentiable, we have

$$0 = \lim_{z \to z_0} \frac{R(z)}{z - z_0} = \lim_{z \to z_0} \frac{R_1(z) + iR_2(z)}{z - z_0}$$

So

$$\lim_{z \to z_0} \frac{R_1(z)}{z - z_0} = \lim_{z \to z_0} \frac{R_2(z)}{z - z_0} = 0$$

Hence, u and v are differentiable, so

$$\frac{\partial u}{\partial x}(z_0) = a, \frac{\partial v}{\partial x}(z_0) = b, \frac{\partial u}{\partial y}(z_0) = -b, \frac{\partial v}{\partial y}(z_0) = a$$

Theorem 2.2 **Cauchy-Riemann Equations**

If $f(z) = u(z) + iv(z)$, then f is differentiable (as a complex function) at z_0 if and only if both u and v are differentiable at z_0 and

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0)$$

and

$$\frac{\partial v}{\partial x}(z_0) = -\frac{\partial u}{\partial y}(z_0)$$

We call these the Cauchy-Riemann Equations.

Recall

Let $u : \mathbb{R}^2 \to \mathbb{R}$. Then u is differentiable at (x_0, y_0) if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ exist in a neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0).

Definition 2.2

The exponential function is the function $f(x + iy) = e^x(\cos y + i \sin y)$.

$$f'(x + iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x(\cos y + i \sin y) = f(x + iy)$$

For $z \in \mathbb{C}$, we write e^z to denote $f(z)$.

Definition 2.3

If we have a function $f : U \to \mathbb{C}$ (with $U \subseteq \mathbb{C}$ open) such that f is differentiable at each point in U, then f is *holomorphic* or *analytic*.

Definition 2.4

$f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is holomorphic on all of $\mathbb{C}$.

2.3 Exponentials, Logarithm Branches, and Polynomial Examples

Remark

For $y \in \mathbb{R}$, we have $e^{iy} = \cos y + i \sin y$. One can see this by expanding the Taylor series for e^{iy} into the Taylor series of $\cos(y)$ and $\sin(y)$.

Definition 2.5 **Logarithm**

A complex number w is a logarithm of $z \in \mathbb{C}$ if $e^w = z$.

Remark

The range of the exponential function is $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. So there is no logarithm of 0.

Definition 2.6

If $G \subseteq \mathbb{C}$ is a non-empty open set not containing 0, then a *branch of log* is a continuous function $\ell : G \to \mathbb{C}$ such that for all $z \in G$, $\ell(z)$ is a logarithm of z.

Note

A branch of log may or may not exist depending on the shape of G.

Example

$G = \mathbb{C} \setminus (-\infty, 0]$. $e^{x+iy} = e^x(\cos y + i \sin y)$. Set $l(z) = \ln(|z|) + i \operatorname{Arg}(z)$. Then

$$\begin{aligned} e^{l(z)} &= e^{\ln(|z|)}(\cos(\operatorname{Arg}(z)) + i \sin(\operatorname{Arg}(z))) \\ &= |z|(\cos(\operatorname{Arg}(z)) + i \sin \operatorname{Arg}(z)) \\ &= z \end{aligned}$$

This is the *principal branch of log*, denoted by $\operatorname{Log}(z)$.

Remark

1. If we defined $l(z) = \operatorname{Log}(x + iy) + i2\pi k$ where $k \in \mathbb{Z}$, then we get a branch of log.
2. It is not possible to find a branch of log on $\mathbb{C}^*$.

Exercise 2.3.1

In polar form, the equations in Cauchy-Riemann Equations (2.2) become: if $f(z) = u(z) + iv(z)$, then $ru_r = v_\theta$ and $u_\theta = -rv_r$.

Theorem 2.3 **Holomorphicity of Logarithm Branches**

If $\ell : G \to \mathbb{C}$ is a branch of log, then ℓ is holomorphic and $\ell'(z) = \frac{1}{z}$.

Proof: Fix $z_0 \in G$. Since ℓ is continuous, as $z \to z_0$ we have

$$w := \ell(z) - \ell(z_0) \to 0.$$

Because $e^{\ell(z)} = z$ and $e^{\ell(z_0)} = z_0$, for $z \neq z_0$,

$$\frac{\ell(z) - \ell(z_0)}{z - z_0} = \frac{w}{e^{\ell(z_0)}(e^w - 1)} = \frac{1}{z_0} \cdot \frac{w}{e^w - 1}.$$

The power series for the exponential gives

$$\lim_{w \to 0} \frac{e^w - 1}{w} = 1.$$

Hence $\ell'(z_0) = \frac{1}{z_0}$. Since z_0 was arbitrary, ℓ is holomorphic on G. □

Theorem 2.4 **Holomorphic Polynomial Factorization Criterion**

Let $P \in \mathbb{C}[x, y]$ and define $F : \mathbb{C} \to \mathbb{C}$, $F(z) = P(\mathrm{Re}(z), \mathrm{Im}(z))$. Then F is holomorphic if and only if there exists $g \in \mathbb{C}[z]$ such that $P(x, y) = g(x + iy)$ for all $(x, y) \in \mathbb{R}^2$.

Proof: Write $f(x, y) = u(x, y) + iv(x, y)$, $f_x = u_x + iv_x$, $f_y = u_y + iv_y$. By Cauchy-Riemann Equations (2.2), $if_x = f_y$. If we write $f(x, y) = \sum_{k=0}^{K} f_k(x, y)$, where f_k is the terms where the sum of the exponents is k.
<u>*Claim :*</u> f satisfies Cauchy-Riemann iff each f_k does.
So, we assume $f(x, y) = c_0 x^n + c_1 x^{n-1} y + \ldots c_n y^n$ for $c_0, c_1, \ldots, c_n \in \mathbb{C}$. Then by CR, $c_k = i^k (n, k) c_0$. So $f(x, y) = c_0 (x + iy)^n = c_0 z^n$. □

3 Series and Analytic Functions

3.1 Numerical Series and Function Series

Definition 3.1 **Series**

A *series* is a formal sum

$$\sum_{n=0}^{\infty} c_n$$

where the terms $c_n \in \mathbb{C}$. Define $S_N = \sum_{k=0}^{N} c_k \in \mathbb{C}$ to be the Nth partial sum of the series. We say a series converges to S if $(S_N)_{N=1}^{\infty}$ converges to S. If it does not converge, then it diverges.

Lemma 3.1 **Terms of a Convergent Series Vanish**

If $\sum_{n=0}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n = 0$.

Proof:

$$0 = \lim_{N\to\infty} (S_N - S_{N-1}) = \lim_{n\to\infty} c_n$$

□

Definition 3.2

A series $\sum_{n=0}^{\infty} c_n$ converges absolutely if $\sum_{n=0}^{\infty} |c_n| < \infty$.

Lemma 3.2 **Absolute Convergence Implies Convergence**

If $\sum_{n=0}^{\infty} c_n$ converges absolutely, then $\sum_{n=0}^{\infty} c_n$ converges.

Proof: Suppose $N < M$.

$$|S_M - S_N| = \left|\sum_{n=N+1}^{M} c_n\right| \le \sum_{n=N+1}^{M} |c_n| \le \sum_{n=N+1}^{\infty} |c_n| \underset{N\to\infty}{\to} 0$$

Therefore, $(S_N)_{N=1}^{\infty}$ is Cauchy. Since $\mathbb{C}$ is complete, then $(S_N)_{N=1}^{\infty}$ converges. □

Note

$\left|\sum_{n=0}^{N} c_n\right| \le \sum_{n=0}^{N} |c_n|$, so if $\sum_{n=0}^{\infty} c_n$ converges absolutely, then

$$\left|\sum_{n=0}^{\infty} c_n\right| \le \sum_{n=0}^{\infty} |c_n| \text{ (by continuity of } |\cdot|\text{)}$$

Example

For $c \in \mathbb{C}$, $\sum_{n=0}^{\infty} c^n$ converges to $\frac{1}{1-c}$ if and only if $|c| < 1$.

Definition 3.3

Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of $\mathbb{C}$-valued functions on $G \subseteq \mathbb{C}$.

1. $(f_n)_{n=1}^{\infty}$ converges pointwise to f on $S \subseteq G$ if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in S$.
2. $(f_n)_{n=1}^{\infty}$ converges uniformly to f on S if
$$\sup_{z\in S} |f_n(z) - f(z)| \underset{n\to\infty}{\to} 0$$
3. $(f_n)_{n=1}^{\infty}$ converges *locally uniformly* to f on G if for each $z \in G$, there is some open ball B with center z such that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on B.

Example

$f_n(z) = z^n$, $G = \{z \in \mathbb{C} \mid |z| < 1\}$. Then $(f_n)_{n=0}^{\infty}$ converges locally uniformly to 0 on G, but does not converge uniformly on G.

Definition 3.4

If $(f_n)_{n=1}^{\infty}$ is a sequence of functions, $\sum_{n=0}^{\infty} f_n(z)$ is the associated series of functions. The series converges to f if the partial sums do.
We say that a series *represents* a function $g : G \to \mathbb{C}$ if for each $z \in G$, the series $\sum_{n=1}^{\infty} f_n(z)$ converges to $g(z)$.

3.2 Power-Series Convergence and Differentiation

Example

$z_0 \in \mathbb{C}$. $(f_n)_{n=0}^{\infty}$ with $f_n(z) = a_n(z - z_0)^n$. We get the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, which we call the *power series centered at* z_0.

Proposition 3.3 **Radial Absolute Convergence of Power Series**

If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges, then for $c \in \mathbb{C}$ with $|c-z_0| < |z-z_0|$, we have that $\sum_{n=0}^{\infty} a_n(c-z_0)^n$ converges absolutely.

Proof: WLOG, suppose $z_0 = 0$. Then $\sum_{n=0}^{\infty} a_n z^n$ converges. Since $\sum_{n=0}^{\infty} a_n z^n$ converges, then $a_n z^n \to 0$. Then $M := \max\{|a_n z^n| \mid n \geq 0\} < \infty$. Let $c \in \mathbb{C}$ with $|c| < |z|$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n c^n| &= \sum_{n=0}^{\infty} \left|a_n z^n \frac{c^n}{z^n}\right| \\
&= \sum_{n=0}^{\infty} |a_n z^n| \left|\frac{c}{z}\right|^n \\
&\leq \sum_{n=0}^{\infty} M \left|\frac{c}{z}\right|^n (*) \\
&< \infty
\end{aligned}$$

since $(*)$ is a geometric series in $\mathbb{R}$ with $\left|\frac{c}{z}\right| < 1$. So $\sum_{n=0}^{\infty} a_n c^n$ converges absolutely by the comparison test in $\mathbb{R}$. □

Corollary 3.4 **Disc of Convergence**

The region of convergence of a power series centered at $z_0 \in \mathbb{C}$ is its open disc of convergence together with possibly some points on the boundary.

Definition 3.5 **Radius of Convergence**

Given $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, let

$$R = \left\{ |c-z_0| : \sum_{n=0}^{\infty} a_n(c-z_0)^n \text{ converges, where } c \in \mathbb{C} \right\}$$

We say that the *radius of convergence* of the power series is $\sup(R)$ if R is bounded, and ∞ otherwise.

Corollary 3.5 **Convergence Inside and Divergence Outside the Radius**

If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R, then for $c \in \mathbb{C}$, $\sum_{n=0}^{\infty} a_n(c-z_0)^n$ converges whenever $|c-z_0| < R$ and diverges whenever $|c-z_0| > R$.

Example

$\sum_{n=0}^{\infty} z^n$ has radius of convergence 1. This represents the function $\frac{1}{1-z}$ on $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. The partial sums are

$$\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}$$

This converges locally uniformly to $\frac{1}{1-z}$.

Theorem 3.6 **Coefficient Comparison for Radii of Convergence**

Let $S_1 := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $S_2 := \sum_{n=0}^{\infty} b_n(z-z_0)^n$ be power series with radius of convergence R_1 and R_2, respectively. If $|b_n| \leq M|a_n|$ for some $M > 0$, then $R_1 \leq R_2$.

Proof: Let $c \in \mathbb{C}$ with $|c - z_0| < R_1$. So S_1 converges absolutely at $z = c$. We have

$$\begin{aligned}\sum_{n=0}^{\infty}|a_n(c-z_0)^n| &= \sum_{n=0}^{\infty}|a_n||c-z_0|^n \\ &\geq \sum_{n=0}^{\infty}\frac{|b_n|}{M}|c-z_0|^n \quad \text{(by comparison test in } \mathbb{R}\text{)} \\ &= \frac{1}{M}\sum_{n=0}^{\infty}|b_n||c-z_0|^n \\ &= \frac{1}{M}\sum_{n=0}^{\infty}|b_n(c-z_0)^n|\end{aligned}$$

So S_2 converges absolutely at $z = c$. Hence, S_2 converges within R_1, so $R_2 \geq R_1$. □

Theorem 3.7 **Cauchy-Hadamard Theorem**

The radius of convergence for a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is

$$\frac{1}{\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

where $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$, and

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup_{m\geq n} a_m$$

Proof: Let

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \in [0,\infty].$$

We show that this number is the radius of convergence.

If $R < \infty$, choose any $r > R$. Then

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > \frac{1}{r},$$

so

$$|a_n| r^n > 1$$

for infinitely many indices n. Hence, for any z with $|z - z_0| = r$, the terms

$$a_n (z - z_0)^n$$

do not approach 0. Thus the power series diverges for every z with $|z - z_0| = r$. Since $r > R$ was arbitrary, the power series diverges for any z with $|z - z_0| > R$.

If $R > 0$, choose any $0 < r < R$. Then

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < \frac{1}{r},$$

so

$$|a_n| r^n < 1$$

for all but finitely many indices n. Hence there exists $M > 0$ such that

$$|a_n| r^n \leq M$$

for every $n \geq 0$. Therefore, for any z with $|z - z_0| < r$,

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| |z - z_0|^n &= \sum_{n=0}^{\infty} |a_n| r^n \left|\frac{z - z_0}{r}\right|^n \\ &\leq M \sum_{n=0}^{\infty} \left|\frac{z - z_0}{r}\right|^n \\ &< \infty.\end{aligned}$$

So the power series converges absolutely for every z with $|z - z_0| < r$. Since $r < R$ was arbitrary, the power series converges for every z with $|z - z_0| < R$.

It follows that the radius of convergence is R. □

Theorem 3.8 **Termwise Differentiation of Power Series**

Suppose $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ has radius of convergence $R > 0$. Let f be the function it represents. Then the radius of convergence for $\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ is also R, and if g is the function it represents, then f is differentiable and $f'(z) = g(z)$ for $z \in G = \{z \in \mathbb{C} \mid |z - z_0| < R\}$.

Proof: Assume WLOG that $z_0 = 0$. Let

$$L := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then $R = \frac{1}{L}$ by Cauchy-Hadamard Theorem (3.7). For the differentiated series, write $b_m = (m+1)a_{m+1}$, so that

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{m=0}^{\infty} b_m z^m.$$

Since $(m+1)^{\frac{1}{m}} \to 1$ and shifting a sequence does not change its limsup,

$$\limsup_{m\to\infty} |b_m|^{\frac{1}{m}} = \limsup_{m\to\infty} \left((m+1)|a_{m+1}|\right)^{\frac{1}{m}} = L.$$

Hence the differentiated series also has radius of convergence $\frac{1}{L} = R$.

Now fix $w \in \mathbb{C}$ with $|w| < R$. Choose ρ such that $|w| < \rho < R$. Then both

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n z^{n-1}$$

converge uniformly on $|z| \le \rho$ by comparison with a convergent geometric series.

For $z \ne w$ with $|z| \le \rho$, we have

$$\frac{f(z) - f(w)}{z - w} = \sum_{n=1}^{\infty} a_n \frac{z^n - w^n}{z - w} = \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} z^k w^{n-1-k}.$$

This series converges uniformly for $|z| \le \rho$, since

$$\left| a_n \sum_{k=0}^{n-1} z^k w^{n-1-k} \right| \le n|a_n|\rho^{n-1},$$

and $\sum_{n=1}^{\infty} n|a_n|\rho^{n-1}$ converges. Hence we may pass the limit $z \to w$ through the sum:

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w} = \sum_{n=1}^{\infty} a_n n w^{n-1} = g(w).$$

Therefore f is differentiable at w and $f'(w) = g(w)$.

Since w was arbitrary with $|w| < R$, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

for all $z \in \{z \in \mathbb{C} \mid |z| < R\}$. □

Example

$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $f'(z) = f(z)$, $f(0) = 1$. Therefore, $f(z) = e^z$.

Corollary 3.9 **Smoothness of Power-Series Sums**

If $f : \mathbb{C} \to \mathbb{C}$ is represented by a power series, then f is smooth.

Corollary 3.10 **Cauchy Product of Power Series**

Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

have radii of convergence R_1 and R_2, respectively. Define

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Then the power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has radius of convergence at least $\min\{R_1, R_2\}$ and represents the product $f(z)g(z)$ whenever $|z - z_0| < \min\{R_1, R_2\}$.

Proof: Fix z with $|z - z_0| < \min\{R_1, R_2\}$. Both series defining $f(z)$ and $g(z)$ converge absolutely. Hence their Cauchy product also converges absolutely, and

$$\begin{aligned} f(z)g(z) &= \left(\sum_{k=0}^{\infty} a_k (z - z_0)^k \right) \left(\sum_{j=0}^{\infty} b_j (z - z_0)^j \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} (z - z_0)^n \\ &= \sum_{n=0}^{\infty} c_n (z - z_0)^n. \end{aligned}$$

This holds for every z satisfying $|z - z_0| < \min\{R_1, R_2\}$. Therefore, the radius of convergence of the Cauchy-product series is at least $\min\{R_1, R_2\}$, and its sum is fg on that disc. □

4 Contour Integration and Local Cauchy Theory

4.1 Complex-Valued Integration

Definition 4.1 **Piecewise Continuity**

Let I be an interval in $\mathbb{R}$ with endpoints $a < b$. A function $\varphi : I \to \mathbb{C}$ is *piecewise continuous* if there exist points

$$a = s_0 < s_1 < \dots < s_n = b$$

with $I \subseteq [a, b]$ and continuous functions $g_j : \left[s_{j-1}, s_j\right] \to \mathbb{C}$ for $j = 1, \dots, n$ such that

$$\varphi(t) = g_j(t) \quad \text{for all} \quad t \in I \cap \left(s_{j-1}, s_j\right).$$

Definition 4.2 **Path**

A *path* is a function $\gamma : [a, b] \to \mathbb{C}$.

Definition 4.3 **Piecewise $\mathcal{C}^1$**

A path $\varphi : [a, b] \to \mathbb{C}$ is *piecewise* $\mathcal{C}^1$ if there is a partition

$$a = t_0 < t_1 < \dots < t_n = b$$

such that $\varphi|_{[t_{j-1}, t_j]}$ is $\mathcal{C}^1$ for every $1 \leq j \leq n$, where derivatives at the endpoints of each subinterval are understood as one-sided derivatives. In particular, φ is continuous on $[a, b]$, while φ' may fail to exist at the interior partition points; at each such point, its one-sided limits exist.

Definition 4.4 **Integral of a Complex Valued Function**

Let $\varphi : [a, b] \to \mathbb{C}$ be piecewise continuous. Then, $\operatorname{Re}(\varphi)$ and $\operatorname{Im}(\varphi)$ are also piecewise continuous. Define

$$\int_a^b \varphi(t) dt := \int_a^b \operatorname{Re}(\varphi) dt + i \int_a^b \operatorname{Im}(\varphi) dt$$

Note

The integral above is well-defined because a piecewise continuous function φ has only finitely many possible discontinuities, namely at the partition points $s_0, \dots, s_n$. Moreover, each g_j is continuous on the compact interval $\left[s_{j-1}, s_j\right]$, hence bounded. Since there are only finitely many g_j's, φ is bounded. The same applies to $\operatorname{Re}(\varphi)$ and $\operatorname{Im}(\varphi)$, so both are bounded with finitely many discontinuities, hence Riemann integrable.

Theorem 4.1 Fundamental Theorem of Calculus (Complex-Valued Functions)

Let $\varphi : [a, b] \to \mathbb{C}$ be continuous and piecewise C^1 on $[a, b]$. Then

$$\int_a^b \varphi'(t)dt = \varphi(b) - \varphi(a)$$

Proof: Write $\varphi(t) = u(t) + iv(t)$, where $u, v : [a, b] \to \mathbb{R}$. Since φ is continuous and piecewise C^1, the functions u and v are continuous and piecewise C^1. Moreover,

$$\varphi'(t) = u'(t) + iv'(t)$$

wherever φ' is defined.

By the definition of the complex-valued integral and the real fundamental theorem of calculus,

$$\begin{aligned} \int_a^b \varphi'(t)dt &= \int_a^b u'(t)dt + i\int_a^b v'(t)dt \\ &= (u(b) - u(a)) + i(v(b) - v(a)) \\ &= (u(b) + iv(b)) - (u(a) + iv(a)) \\ &= \varphi(b) - \varphi(a). \end{aligned}$$

□

Theorem 4.2 Triangle Inequality for Complex-Valued Integrals

Let $\varphi : [a, b] \to \mathbb{C}$ be piecewise continuous. Then

$$\left| \int_a^b \varphi(t)dt \right| \leq \int_a^b |\varphi(t)|dt.$$

Proof: If $\int_a^b \varphi(t)dt = 0$, then the result is immediate. Otherwise, define

$$\lambda := \frac{\left|\int_a^b \varphi(t)dt\right|}{\int_a^b \varphi(t)dt}.$$

Then $|\lambda| = 1$ and

$$\begin{aligned}
\left|\int_a^b \varphi(t)dt\right| &= \lambda \int_a^b \varphi(t)dt \\
&= \int_a^b \lambda\varphi(t)dt \\
&= \mathrm{Re}\left(\int_a^b \lambda\varphi(t)dt\right) \\
&= \int_a^b \mathrm{Re}(\lambda\varphi(t))dt \\
&\le \int_a^b |\lambda\varphi(t)|dt \\
&= \int_a^b |\varphi(t)|dt.
\end{aligned}$$

□

4.2 Path Integrals, Reparametrization, and the ML Inequality

Definition 4.5 — **Path Integral**

Let $\gamma : [0, 1] \to G$ be a piecewise-$\mathcal{C}^1$ path, and suppose $f : G \to \mathbb{C}$ is continuous on a subset of G containing the range of γ. Define

$$\int_\gamma f(z)dz := \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

Proposition 4.3 — **Invariance Under Reparametrization**

Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise-$\mathcal{C}^1$, and let $\beta : [a_1, b_1] \to [a, b]$ be increasing and piecewise-$\mathcal{C}^1$ with $\beta(a_1) = a$ and $\beta(b_1) = b$. Set $\gamma_1 = \gamma \circ \beta$. If f is continuous on the range of γ, then

$$\begin{aligned}
\int_{\gamma_1} f(z)dz &= \int_{a_1}^{b_1} f(\gamma_1(s)){\gamma_1}'(s)ds \\
&= \int_{a_1}^{b_1} f(\gamma(\beta(s)))\gamma'(\beta(s))\beta'(s)ds \\
&= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_\gamma f(z)dz.
\end{aligned}$$

Theorem 4.4 **Fundamental Theorem of Calculus (For Path Integrals)**

If $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$, and if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on an open set that contains the range of γ, then

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

Proof:

$$\begin{aligned}\int_\gamma f'(z)dz &= \int_a^b f'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b (f \circ \gamma)'(t)dt \\ &= (f \circ \gamma)(b) - (f \circ \gamma)(a) \\ &= f(\gamma(b)) - f(\gamma(a)).\end{aligned}$$

□

Note **Additivity Under Subdivision**

Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise-$\mathcal{C}^1$, let $c \in [a, b]$, and define

$$\gamma_1 = \gamma|_{[a,c]}, \quad \gamma_2 = \gamma|_{[c,b]}.$$

If f is continuous on the range of γ, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

Example

Let $z_0 \in \mathbb{C}$, $R > 0$, and let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = z_0 + Re^{it}$. If $n \in \mathbb{Z}$ and $n \neq -1$, then

$$\int_\gamma (z - z_0)^n dz = 0.$$

Indeed, $f(z) = \frac{(z-z_0)^{n+1}}{n+1}$ satisfies $f'(z) = (z - z_0)^n$, so by Fundamental Theorem of Calculus (For Path Integrals) (4.4),

$$\int_\gamma (z - z_0)^n dz = f(\gamma(2\pi)) - f(\gamma(0)) = 0.$$

For $n = -1$, we have

$$\int_\gamma \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{1}{Re^{it}} iRe^{it} dt$$
$$= i \int_0^{2\pi} dt$$
$$= 2\pi i.$$

Definition 4.6 **Length of a Path**

Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise-$\mathcal{C}^1$. The length of γ is

$$L(\gamma) := \int_a^b |\gamma'(t)| dt.$$

Theorem 4.5 **ML Inequality**

Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise-$\mathcal{C}^1$, and suppose f is continuous on the range of γ. If

$$M := \max_{t \in [a,b]} |f(\gamma(t))|,$$

then

$$\left| \int_\gamma f(z) dz \right| \leq ML(\gamma).$$

Proof:

$$\left| \int_\gamma f(z) dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t) dt \right|$$
$$\leq \int_a^b |f(\gamma(t))\gamma'(t)| dt$$
$$= \int_a^b |f(\gamma(t))||\gamma'(t)| dt$$
$$\leq \int_a^b M|\gamma'(t)| dt$$
$$= ML(\gamma).$$

$\square$

Theorem 4.6 **Interchange of Limit and Path Integral**

Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise-$\mathcal{C}^1$, and let $(f_n)_{n=1}^{\infty}$ be continuous on the range of γ. If $f_n \to f$ uniformly on the range of γ, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

Proof: For each n, by ML Inequality (4.5) applied to $f_n - f$,

$$\begin{aligned}\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| &= \left|\int_\gamma (f_n - f)(z)dz\right| \\ &\le L(\gamma) \max_{t\in[a,b]} |f_n(\gamma(t)) - f(\gamma(t))|.\end{aligned}$$

The maximum on the right tends to 0 by uniform convergence on the range of γ. Hence the integrals converge to $\int_\gamma f(z)dz$. $\square$

4.3 Goursat's Lemma: Cauchy's Theorem for Triangles

Example

Let $R_{a,b}$ be the rectangle with vertices $-a + ib, a + ib, a - ib, -a - ib$, oriented counter-clockwise. We want to compute

$$\int_{R_{a,b}} e^{-z^2}\,\mathrm{d}z.$$

Since

$$e^{-z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!},$$

define

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!}.$$

Then, by Termwise Differentiation of Power Series (3.8),

$$g'(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!} = e^{-z^2}.$$

Therefore

$$\int_{R_{a,b}} e^{-z^2}\,\mathrm{d}z = \int_{R_{a,b}} g'(z)\,\mathrm{d}z = 0.$$

Equivalently, if $\gamma_1, \ldots, \gamma_4$ are the four edges of the rectangle, then

$$\sum_{j=1}^{4} \int_{\gamma_j} e^{-z^2}\, \mathrm{d}z = 0.$$

One can compute the four edge integrals separately and use this identity in the Fourier transform of the Gaussian.

Theorem 4.7 **Cauchy's Theorem for Triangles (Goursat's Lemma)**

Let Δ be a closed solid triangle in $\mathbb{C}$, and let T be a closed path which goes once (counter-clockwise) around the boundary of Δ. If f is holomorphic on an open set containing Δ, then

$$\int_T f(z) dz = 0.$$

Proof: Let

$$I = \int_T f(z) dz.$$

Set $\Delta_0 = \Delta$ and $T_0 = T$. Split Δ_0 into four closed solid triangles by joining the midpoints of its sides, and let $S_1, \ldots, S_4$ be their counter-clockwise boundary loops.

When the four boundary integrals are added, every interior edge is traversed once in each direction. Hence the integrals over the interior edges cancel, while the remaining outer edges give exactly the integral over T. Therefore

$$I = \sum_{j=1}^{4} \int_{S_j} f(z) dz.$$

By the triangle inequality, at least one of these four integrals has absolute value at least $\frac{|I|}{4}$. Thus there is some $j_1 \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{S_{j_1}} f(z) dz \right| \geq \frac{|I|}{4}.$$

Let Δ_1 be the corresponding subtriangle and let $T_1 = S_{j_1}$.

Now repeat the same construction inside the selected subtriangle, and then continue inductively. This gives nested closed solid triangles

$$\Delta_0 \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \ldots$$

with counter-clockwise boundary loops $T_0, T_1, T_2, \ldots$ such that

$$\left|\int_{T_n} f(z)dz\right| \geq \frac{|I|}{4^n},$$

while

$$L(T_n) = 2^{-n}L(T), \quad \mathrm{diam}(\Delta_n) = 2^{-n}\mathrm{diam}(\Delta).$$

The length and diameter identities hold because each subdivision replaces the chosen triangle by a similar triangle whose side lengths are half as large.

Note that Δ is compact, and for each $n \in \mathbb{N}$, Δ_n is closed in the subspace topology of Δ (since Δ_n is closed in $\mathbb{C}$). Note further that Δ_n is non-empty with $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_n$. Therefore, $\bigcap_{n=1}^{\infty} \Delta_n \neq \varnothing$, so there exists $z_0 \in \bigcap_{n=0}^{\infty} \Delta_n$. Since f is holomorphic at z_0, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

where $|\eta(z)| \leq \varepsilon$ whenever $|z - z_0| < \delta$. Choose n large enough that $\Delta_n \subseteq B_\delta(z_0)$; this is possible because $z_0 \in \Delta_n$ and $\mathrm{diam}(\Delta_n) \to 0$.

On the loop T_n, the expansion above gives

$$\int_{T_n} f(z)dz = \int_{T_n} f(z_0)dz + \int_{T_n} f'(z_0)(z - z_0)dz + \int_{T_n} \eta(z)(z - z_0)dz.$$

The first two integrals are 0 because the functions $f(z_0)$ and $f'(z_0)(z - z_0)$ have complex primitives $f(z_0)z$ and $f'(z_0)\frac{(z-z_0)^2}{2}$, and T_n is closed. Hence

$$\int_{T_n} f(z)dz = \int_{T_n} \eta(z)(z - z_0)dz.$$

Since $|\eta(z)| \leq \varepsilon$ and $\Delta_n \subseteq B_\delta(z_0)$ By ML Inequality (4.5),

$$\begin{aligned}\left|\int_{T_n} f(z)dz\right| &= \left|\int_{T_n} \eta(z)(z - z_0)dz\right| \\ &\leq \varepsilon \ \mathrm{diam}(\Delta_n)L(T_n) \\ &= \varepsilon 4^{-n} \ \mathrm{diam}(\Delta)L(T).\end{aligned}$$

Here we used that every point of T_n lies in Δ_n, so $|z - z_0| \leq \mathrm{diam}(\Delta_n)$. Combining this with

$$|I| \leq 4^n \left|\int_{T_n} f(z)dz\right|$$

gives

$$|I| \leq \varepsilon \ \mathrm{diam}(\Delta)L(T).$$

Since $\varepsilon > 0$ was arbitrary, $I = 0$. □

4.4 Cauchy's Theorem and Formula on Discs

Theorem 4.8 **Cauchy's Theorem for Convex Regions**

Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in $G \subseteq \mathbb{C}$ and G is *convex* (if $z_1, z_2 \in G$, then $[z_1, z_2] \in G$), then for any piecewise $\mathcal{C}'$ closed curve γ in G, we have

$$\int_\gamma f(z)dz = 0$$

Proof: Note that if $f(z) = g'(z)$ with g holomorphic in G, then the theorem follows from Fundamental Theorem of Calculus (For Path Integrals) (4.4) (since γ is closed).
Fix $z_0 \in G$. Define

$$g(z) = \int_{[z_0,z]} f(\zeta)d\zeta.$$

This is well-defined because G is convex, so $[z_0, z] \subseteq G$ for every $z \in G$.

We claim that $g'(z) = f(z)$ for every $z \in G$. Fix $z \in G$ and take $h \in \mathbb{C}$ with $h \neq 0$ and $z + h \in G$. Since G is convex, the triangle with vertices $z_0, z, z + h$ lies in G. By Cauchy's Theorem for Triangles (Goursat's Lemma) (4.7),

$$\int_{[z_0,z]} f(\zeta)d\zeta + \int_{[z,z+h]} f(\zeta)d\zeta + \int_{[z+h,z_0]} f(\zeta)d\zeta = 0.$$

Hence

$$g(z+h) - g(z) = \int_{[z,z+h]} f(\zeta)d\zeta.$$

Therefore

$$\frac{g(z+h) - g(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]} (f(\zeta) - f(z))d\zeta.$$

Parametrize $[z, z + h]$ by $\zeta(t) = z + th$, $0 \le t \le 1$. Then

$$\left|\frac{g(z+h) - g(z)}{h} - f(z)\right| \le \int_0^1 |f(z+th) - f(z)|dt.$$

Since f is continuous at z, the right-hand side tends to 0 as $h \to 0$. Thus $g'(z) = f(z)$.

Now Fundamental Theorem of Calculus (For Path Integrals) (4.4) gives

$$\int_\gamma f(z)dz = \int_\gamma g'(z)dz = g(\gamma(b)) - g(\gamma(a)) = 0,$$

because γ is closed. $\square$

Theorem 4.9 **Cauchy's Formula for the Circle**

Let $\bullet$ be a closed (solid) disc in $\mathbb{C}$, and C be a closed path oriented counter-clockwise around the boundary of $\bullet$. If f is holomorphic on an open set that contains $\bullet$, then if z is in the interior of $\bullet$, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$$

Proof: Let U be an open set containing $\bullet$ on which f is holomorphic. Fix $z_0 \in \operatorname{int}(\bullet)$. Write $\bullet = \{z \in \mathbb{C} \mid |z-a| \le R\}$, so C goes once counter-clockwise around $\{z \in \mathbb{C} \mid |z-a| = R\}$.

Define $h : U \to \mathbb{C}$ by

$$h(\zeta) = \begin{cases} \frac{f(\zeta)-f(z_0)}{\zeta - z_0}, \text{if } \zeta \ne z_0 \\ f'(z_0), \text{if } \zeta = z_0. \end{cases}$$

Then h is continuous on U and holomorphic on $U \setminus \{z_0\}$. By the same argument as in Cauchy's Theorem for Triangles (Goursat's Lemma) (4.7), the one exceptional point z_0 contributes nothing, so

$$\int_C h(\zeta)d\zeta = 0.$$

Hence

$$\int_C \frac{f(\zeta)}{\zeta - z_0}d\zeta = f(z_0)\int_C \frac{1}{\zeta - z_0}d\zeta.$$

It remains to compute the last integral. By Invariance Under Reparametrization (4.3), we may use the standard counter-clockwise parametrization $\gamma(t) = a + Re^{it}$, $0 \le t \le 2\pi$. Since $z_0 \in \operatorname{int}(\bullet)$, $|z_0 - a| < R$. Thus

$$\begin{aligned} \int_C \frac{1}{\zeta - z_0}d\zeta &= \int_0^{2\pi} \frac{iRe^{it}}{a - z_0 + Re^{it}}dt \\ &= i\int_0^{2\pi} \frac{1}{1 + \frac{a-z_0}{Re^{it}}}dt \\ &= i\int_0^{2\pi}\sum_{n=0}^{\infty}\left(-\frac{a-z_0}{R}\right)^n e^{-int}dt \\ &= 2\pi i. \end{aligned}$$

The geometric series above converges uniformly in t because $\frac{|a-z_0|}{R} < 1$, so Interchange of Limit and Path Integral (4.6) justifies interchanging the sum and integral. Therefore

$$\int_C \frac{f(\zeta)}{\zeta - z_0} d\zeta = 2\pi i f(z_0),$$

which is the desired formula. □

Corollary 4.10 **Cauchy Integral Outside the Disc**

Let $\overline{D}(a, R)$ be a closed disc, and let C be its counter-clockwise boundary. Suppose that f is holomorphic on an open set U containing $\overline{D}(a, R)$. If $w \notin \overline{D}(a, R)$, then

$$\int_C \frac{f(\zeta)}{\zeta - w} d\zeta = 0.$$

Proof: Since $\overline{D}(a, R)$ is compactly contained in U and $R < |w - a|$, choose ρ such that $R < \rho < |w - a|$ and $\overline{D}(a, \rho) \subseteq U$. Then $\zeta \mapsto \frac{f(\zeta)}{\zeta - w}$ is holomorphic on the convex disc $D(a, \rho)$, which contains C. The conclusion follows from Cauchy's Theorem for Convex Regions (4.8). □

4.5 Analyticity and Cauchy's Formulas for Derivatives

Corollary 4.11 **Mean Value Property for Holomorphic Functions**

Suppose f is holomorphic on the disc $D(z_0, R)$. Let $0 < r < R$, and let C_r be the circle of radius r centered at z_0, oriented counter-clockwise. By Cauchy's Formula for the Circle (4.9),

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z_0} d\zeta.$$

Parametrize C_r by $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$, so that $\gamma'(t) = ire^{it}$. Then

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt. \end{aligned}$$

Proposition 4.12 **Holomorphic Functions Are Locally Analytic**

Let $G \subseteq \mathbb{C}$ be open, let $f : G \to \mathbb{C}$ be holomorphic, and fix $z_0 \in G$. Choose $r > 0$ such that

$$\overline{D}(z_0, r) \subseteq G,$$

and let C_r be the positively oriented boundary of this closed disc. Then, for every $z \in D(z_0, r)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

In particular, every holomorphic function is locally analytic.

Proof: Fix $z \in D(z_0, r)$. For $\zeta \in C_r$, we have

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \quad (\heartsuit) \text{ (by geometric series formula)}
\end{aligned}$$

since $\left|\frac{z - z_0}{\zeta - z_0}\right| < 1$. Define, for $\zeta \in C_r$,

$$g_N(\zeta) = \sum_{n=0}^{N} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

and

$$g(\zeta) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

<u>*Claim 1:*</u> $g_N \to g$ uniformly on C_r.
Proof: Set $q = \frac{|z - z_0|}{r} < 1$, and let $\varepsilon > 0$. Since the geometric series

$$\sum_{n=0}^{\infty} q^n$$

converges, its tails converge to 0. Thus there exists $N_0 \in \mathbb{N}$ such that, whenever $N \geq N_0$,

$$\sum_{n=N+1}^{\infty} q^n < r\varepsilon.$$

Therefore, for every $\zeta \in C_r$ and every $N \geq N_0$,

$$\begin{aligned} |g(\zeta) - g_N(\zeta)| &\leq \sum_{n=N+1}^{\infty} \frac{|z - z_0|^n}{r^{n+1}} \\ &= \frac{1}{r} \sum_{n=N+1}^{\infty} q^n \\ &< \varepsilon. \end{aligned}$$

Since N_0 is independent of ζ, this proves uniform convergence. □

Claim 2: $fg_N \to fg$ uniformly on C_r.
Proof: Since f is continuous on the compact set C_r, let

$$M_1 = \max\left\{1, \max_{\zeta \in C_r} |f(\zeta)|\right\}.$$

Given $\varepsilon > 0$, Claim 1 gives N_0 such that, whenever $N \geq N_0$,

$$|g_N(\zeta) - g(\zeta)| < \frac{\varepsilon}{M_1}$$

for every $\zeta \in C_r$. Hence

$$|f(\zeta)g_N(\zeta) - f(\zeta)g(\zeta)| \leq M_1 |g_N(\zeta) - g(\zeta)| < \varepsilon,$$

proving Claim 2. □

We may now compute

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - z}d\zeta \quad \text{(Cauchy's Integral Formula)} \\
&= \frac{1}{2\pi i}\int_{C_r} f(\zeta)\left(\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}\right)d\zeta \quad (\text{by } \heartsuit) \\
&= \frac{1}{2\pi i}\int_{C_r} f(\zeta)\left(\lim_{N\to\infty}\sum_{n=0}^{N}\frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}\right)d\zeta \quad \text{(by definition of an infinite sum)} \\
&= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{C_r} f(\zeta)\left(\sum_{n=0}^{N}\frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}\right)d\zeta \quad \text{(by Interchange of limit} \\
&\text{and path integral, since claim 2 holds)} \\
&= \frac{1}{2\pi i}\lim_{N\to\infty}\sum_{n=0}^{N}\int_{C_r} f(\zeta)\frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}d\zeta \quad \text{(by } \mathbb{C}\text{-linearity of the integral)} \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right)(z-z_0)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right)(z-z_0)^n \quad \text{(by definition)} \\
&= \sum_{n=0}^{\infty} a_n (z-z_0)^n
\end{aligned}$$

where $a_n := \left(\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right)$. Since a_n does not depend on z, this is a power-series representation of f on $D(z_0, r)$, centered at z_0. Therefore, f is locally analytic (at z_0). □

Corollary 4.13 **Cauchy's Formula for Derivatives**

Let $G \subseteq \mathbb{C}$ be open, let $f : G \to \mathbb{C}$ be holomorphic, and fix $z_0 \in G$. Choose $r > 0$ such that $\overline{D}(z_0, r) \subseteq G$, and let C_r be the positively oriented boundary of this closed disc. Then, for every integer $n \geq 0$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta.$$

Proof: By Holomorphic Functions Are Locally Analytic (4.12),

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

on $D(z_0, r)$, where

$$a_k = \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta.$$

Differentiating the power series n times term by term gives

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k (z - z_0)^{k-n}.$$

Evaluating at $z = z_0$, every term vanishes except the term with $k = n$. Therefore,

$$\begin{aligned} f^{(n)}(z_0) &= n! a_n \\ &= \frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \end{aligned}$$

where the second equality follows from the coefficient formula above. □

Corollary 4.14 **Holomorphic Functions are Smooth**

If $G \subseteq \mathbb{C}$ is open and $f : G \to \mathbb{C}$ is holomorphic, then f' is holomorphic on G. In particular, every holomorphic function is infinitely differentiable.

Proof: Fix $z_0 \in G$. Since G is open, there is an $r > 0$ such that $\overline{D}(z_0, r) \subseteq G$. By Holomorphic Functions Are Locally Analytic (4.12), f has a power-series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on a neighbourhood of z_0. Differentiating term by term gives

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

The differentiated series has the same radius of convergence, so it defines a holomorphic function near z_0. Thus f' is holomorphic at every point of G, and hence on all of G. Repeating the argument shows that all higher derivatives of f exist and are holomorphic.
□

Example

The converse is false. Let

$$f(x + iy) = \frac{1}{1 + x^2}.$$

Then f is smooth as a function $\mathbb{R}^2 \to \mathbb{C}$. However, if $f = u + iv$, then

$$u(x, y) = \frac{1}{1 + x^2}, \quad v(x, y) = 0.$$

Hence

$$u_x = -2\frac{x}{(1+x^2)^2}, \quad v_y = 0.$$

The equation $u_x = v_y$ from Cauchy-Riemann Equations (2.2) fails whenever $x \neq 0$, so f is not holomorphic on any open subset of $\mathbb{C}$.

4.6 Morera, Liouville, and the Fundamental Theorem of Algebra

Corollary 4.15 **Morera's Theorem**

Let $G \subseteq \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be continuous. Suppose that for every closed solid triangle $\Delta \subseteq G$, oriented counter-clockwise along its boundary $\partial\Delta$, one has

$$\int_{\partial\Delta} f(z)dz = 0.$$

Then f is holomorphic on G.

Proof: Fix $z_0 \in G$. Choose $r > 0$ such that the open disc

$$D = \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

is contained in G. Since D is convex, the line segment $[z_0, z]$ lies in D for every $z \in D$. Define

$$F(z) = \int_{[z_0,z]} f(\zeta)d\zeta.$$

Fix $z \in D$ and take $h \in \mathbb{C}$ such that $h \neq 0$ and $z + h \in D$. The triangle with vertices $z_0, z, z + h$ lies in D, so the hypothesis gives

$$\int_{[z_0,z]} f(\zeta)d\zeta + \int_{[z,z+h]} f(\zeta)d\zeta + \int_{[z+h,z_0]} f(\zeta)d\zeta = 0.$$

Therefore,

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)d\zeta.$$

Parametrizing $[z, z + h]$ by $\zeta(t) = z + th$, $0 \leq t \leq 1$, yields

$$\frac{F(z+h) - F(z)}{h} - f(z) = \int_0^1 (f(z+th) - f(z))dt.$$

Since f is continuous at z, the right-hand side tends to 0 as $h \to 0$. Hence $F'(z) = f(z)$ for every $z \in D$. Thus F is holomorphic on D, and Holomorphic Functions are Smooth (4.14) implies that $f = F'$ is holomorphic on D.

Since z_0 was arbitrary, f is holomorphic on G. □

Theorem 4.16 **Liouville Theorem**

Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded, then f is constant.

Proof: Fix $z_0 \in \mathbb{C}$. Let C_r be the circle with center z_0 and radius r oriented counterclockwise. Then

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta$$

So

$$\begin{aligned} |f'(z_0)| &\leq \frac{1}{2\pi} \left| \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \right| \\ &\leq \frac{1}{2\pi} L(C_r) \sup_{\zeta \in C_r} \left\{ \left| \frac{f(\zeta)}{(\zeta - z_0)^2} \right| \right\} \\ &\leq r \frac{M}{r^2} = \frac{1}{r} M \underset{r \to \infty}{\longrightarrow} 0 \end{aligned}$$

□

Theorem 4.17 **Fundamental Theorem of Algebra**

Every nonconstant polynomial over $\mathbb{C}$ splits into linear factors. More explicitly, if

$$P(z) = \sum_{k=0}^{n} a_k z^k, \quad a_n \neq 0, \quad n \geq 1,$$

then there exist $b_1, \ldots, b_n \in \mathbb{C}$ such that

$$P(z) = a_n \prod_{j=1}^{n} (z - b_j).$$

Proof: We begin with three claims.

Claim 1: $|P(z)| \to \infty$ as $|z| \to \infty$.
Proof of Claim 1: Set

$$M = \sum_{k=0}^{n-1} |a_k|.$$

For $|z| \geq 1$,

$$\left| \sum_{k=0}^{n-1} a_k z^k \right| \leq \sum_{k=0}^{n-1} |a_k| |z|^k \leq M |z|^{n-1}.$$

Hence

$$\begin{aligned}|P(z)| &\geq |a_n||z|^n - \left|\sum_{k=0}^{n-1} a_k z^k\right| \\ &\geq |a_n||z|^n - M|z|^{n-1}.\end{aligned}$$

If $|z| \geq \max\left\{1, 2\frac{M}{|a_n|}\right\}$, then

$$|P(z)| \geq \frac{|a_n|}{2}|z|^n.$$

The right-hand side tends to ∞ as $|z| \to \infty$, which proves the claim. □

Claim 2: If $P(b) = 0$, then there is a polynomial Q such that $P(z) = (z - b)Q(z)$. Moreover, Q has degree $n - 1$ and leading coefficient a_n.

Proof of Claim 2: By polynomial division, there are a polynomial Q and a constant r such that

$$P(z) = (z - b)Q(z) + r.$$

Evaluating at $z = b$ gives $r = P(b) = 0$. Thus

$$P(z) = (z - b)Q(z).$$

Comparing degrees and leading coefficients shows that Q has degree $n - 1$ and leading coefficient a_n. □

Claim 3: P has a zero in $\mathbb{C}$.

Proof: Suppose instead that $P(z) \neq 0$ for every $z \in \mathbb{C}$. Then

$$g : \mathbb{C} \to \mathbb{C}, \quad g(z) = \frac{1}{P(z)}$$

is entire. By Claim 1, there is $R > 0$ such that $|P(z)| \geq 1$ whenever $|z| \geq R$. Consequently, $|g(z)| \leq 1$ outside $\overline{D}(0, R)$.

The function g is continuous on the compact disc $\overline{D}(0, R)$, so it is bounded there as well. Therefore g is bounded on $\mathbb{C}$. By Liouville Theorem (4.16), g is constant, which would force P to be constant. This is a contradiction, so P has a zero in $\mathbb{C}$. □

We now proceed by induction on n. If $n = 1$, Claim 3 gives a zero $b_1 \in \mathbb{C}$. By Claim 2,

$$P(z) = (z - b_1)Q(z),$$

where Q has degree 0 and leading coefficient a_1. Hence $Q = a_1$, and

$$P(z) = a_1 \cdot (z - b_1).$$

Now suppose $n > 1$ and that the result holds for polynomials of degree $n - 1$. By Claim 3, P has a zero $b_1 \in \mathbb{C}$. By Claim 2, there is a polynomial Q of degree $n - 1$ and leading coefficient a_n such that

$$P(z) = (z - b_1)Q(z).$$

By the induction hypothesis, there are $b_2, ..., b_n \in \mathbb{C}$ such that

$$Q(z) = a_n \prod_{j=2}^{n} (z - b_j).$$

Hence

$$P(z) = a_n \prod_{j=1}^{n} (z - b_j),$$

as required. □

5 Zeros, Holomorphic Limits, and Modulus Principles

5.1 Zeros and the Identity Theorem

Definition 5.1 **Zero of Order n or ∞**

Suppose $f : G \to \mathbb{C}$ is holomorphic and $f(z_0) = 0$ for some $z_0 \in G$. We say that z_0 is a *zero of finite order* $m \geq 1$ if

$$f(z_0) = f'(z_0) = ... = f^{(m-1)}(z_0) = 0 \quad \text{but} \quad f^{(m)}(z_0) \neq 0.$$

We say that z_0 is a *zero of order* ∞ if

$$f^{(n)}(z_0) = 0$$

for every $n \geq 0$, where $f^{(0)} = f$.

Thus every zero is either of finite order or of order ∞, and it can have only one order: the finite order, if it exists, is the least $m \geq 1$ such that $f^{(m)}(z_0) \neq 0$. A zero of order 1 is called a *simple zero.*

Proposition 5.1 **Zero of Infinite Order Implies Function is Identically Zero**

Let f be holomorphic on a region G, and suppose that

$$f^{(n)}(z_0) = 0$$

for every $n \geq 0$ at some $z_0 \in G$. Then f is identically zero on G.

Proof: Define

$$S = \{z \in G \mid f^{(n)}(z) = 0 \;\; \text{for every} \;\; n \geq 0\}.$$

By assumption, $z_0 \in S$, so S is nonempty.

To see that S is open, fix $w \in S$. On some disc $B_r(w) \subseteq G$, the Taylor expansion of f at w gives

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n = 0.$$

Thus f vanishes on $B_r(w)$, so every derivative of f also vanishes there. Hence $B_r(w) \subseteq S$.

The set S is closed in G because every derivative $f^{(n)}$ is continuous. Since G is connected and S is nonempty, open, and closed, we have $S = G$. In particular, $f \equiv 0$ on G. □

Proposition 5.2 **Zeros of Finite Order Are Isolated**

Let f be holomorphic on an open set $G \subseteq \mathbb{C}$. If $z_0 \in G$ is a zero of f of finite order m, then there exists $r > 0$ such that

$$f(z) \neq 0$$

whenever $0 < |z - z_0| < r$. Thus z_0 is an *isolated zero* of f.

Proof: On a disc centred at z_0, write the Taylor expansion

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n, \quad a_m \neq 0.$$

Factoring out $(z - z_0)^m$ gives

$$f(z) = (z - z_0)^m g(z), \quad g(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m}.$$

The function g is holomorphic near z_0 and $g(z_0) = a_m \neq 0$. Therefore, by continuity, there is an $r > 0$ such that $g(z) \neq 0$ whenever $|z - z_0| < r$. Hence $f(z) \neq 0$ whenever $0 < |z - z_0| < r$. □

Theorem 5.3 **Identity Theorem**

Let G be a region. If $f, g : G \to \mathbb{C}$ are holomorphic, and if f and g agree on a set with a limit point in G, then $f = g$ on G.

Proof: Let $(z_n)_{n=1}^{\infty}$ be a sequence of distinct points in G such that $z_n \to z_0 \in G$ and $f(z_n) = g(z_n)$ for every n. Set $h = f - g$. Then h is holomorphic on G, and

$$h(z_n) = 0 \quad \text{for every } n \in \mathbb{N}.$$

By continuity, $h(z_0) = 0$ as well.

Since $z_n \to z_0$ and the z_n are distinct, every neighbourhood of z_0 contains some zero of h other than z_0. Thus z_0 is not isolated, so by Zeros of Finite Order Are Isolated (5.2), it is not a zero of finite order.

Since every zero has either finite order or order ∞, z_0 is a zero of infinite order of h. By Zero of Infinite Order Implies Function is Identically Zero (5.1), we have $h \equiv 0$ on G. Therefore $f = g$ on G. □

Example

Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2}$$

for every $n \geq 2$. We can determine $f'\left(\frac{i}{2}\right)$. Let $g(z) = z^2$. Then $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for every $n \geq 2$, and $\frac{1}{n} \to 0 \in \mathbb{D}$. Since the set $\left\{\frac{1}{n} \mid n \geq 2\right\}$ has a limit point in $\mathbb{D}$, Identity Theorem (5.3) gives $f = g$ on $\mathbb{D}$. Therefore,

$$f'\left(\frac{i}{2}\right) = g'\left(\frac{i}{2}\right) = 2\left(\frac{i}{2}\right) = i.$$

5.2 Locally Uniform Limits of Holomorphic Functions

Theorem 5.4 **Weierstrass Convergence Test**

Let $G \subseteq \mathbb{C}$ be open, and let $f_k : G \to \mathbb{C}$ be holomorphic for every $k \in \mathbb{N}$. If $f_k \to f$ locally uniformly on G, then f is holomorphic on G. Moreover, for every $n \in \mathbb{N}$, $f_k^{(n)} \to f^{(n)}$ locally uniformly on G.

Example **Weierstrass Convergence Test fails in $\mathbb{R}$**

The analogous statement is false for real differentiability. For example, define

$$f_k(x) = \sqrt{x^2 + \frac{1}{k}}, \quad x \in \mathbb{R}.$$

Each f_k is smooth on $\mathbb{R}$, and $f_k \to |x|$ uniformly on every compact interval. However, $|x|$ is not differentiable at 0.

Proof of Weierstrass Convergence Test: Since each f_k is continuous and $f_k \to f$ locally uniformly, f is continuous. Let $\Delta \subseteq G$ be any closed solid triangle. The convergence is uniform on the compact curve $\partial\Delta$, so

$$\int_{\partial\Delta} f(z)dz = \lim_{k\to\infty} \int_{\partial\Delta} f_k(z)dz = 0.$$

Here the first equality follows from Interchange of Limit and Path Integral (4.6), and the second follows from Cauchy's Theorem for Triangles (Goursat's Lemma) (4.7). Therefore, by Morera's Theorem (4.15), f is holomorphic on G.

Fix $z_0 \in G$ and choose $0 < r < R$ such that

$$\overline{D}(z_0, R) \subseteq G.$$

Set $\rho = R - r > 0$, and fix $n \in \mathbb{N}$. For every $z \in \overline{D}(z_0, r)$, we have $\overline{D}(z, \rho) \subseteq \overline{D}(z_0, R)$. Applying the coefficient formula in Holomorphic Functions Are Locally Analytic (4.12) to

$f_k - f$ on $C_\rho(z)$, and using that the n-th derivative at the center is $n!$ times the n-th coefficient, gives

$$\left|f_k^{(n)}(z) - f^{(n)}(z)\right| \leq \frac{n!}{\rho^n} \sup_{\zeta \in C_\rho(z)} |f_k(\zeta) - f(\zeta)| \leq \frac{n!}{\rho^n} \sup_{\zeta \in \overline{D}(z_0, R)} |f_k(\zeta) - f(\zeta)|.$$

The right-hand side is independent of z and tends to 0 by local uniform convergence. Hence $f_k^{(n)} \to f^{(n)}$ uniformly on $\overline{D}(z_0, r)$. Since z_0 and n were arbitrary, the convergence of every derivative is locally uniform on G. □

5.3 Maximum and Minimum Modulus Principles; Schwarz's Lemma

Theorem 5.5 **Maximum Modulus Principle**

If G is a region and $f : G \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no local maximum in G.

Proof: Suppose, toward a contradiction, that $|f|$ has a local maximum at $z_0 \in G$. Choose $r > 0$ such that

$$\overline{D}(z_0, r) \subseteq G$$

and

$$|f(z)| \leq |f(z_0)|$$

for every $z \in \overline{D}(z_0, r)$.

If $f(z_0) = 0$, then $f = 0$ on $D(z_0, r)$. By Identity Theorem (5.3), f is identically zero on G, contrary to the assumption that f is non-constant. Hence $f(z_0) \neq 0$.

Fix $0 < \rho < r$. By Mean Value Property for Holomorphic Functions (4.11),

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it}) dt.$$

Dividing by $f(z_0)$ gives

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{f(z_0)} dt.$$

Taking real parts,

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{f(z_0 + \rho e^{it})}{f(z_0)}\right) dt.$$

But

$$\operatorname{Re}\left(\frac{f(z_0 + \rho e^{it})}{f(z_0)}\right) \leq \left|\frac{f(z_0 + \rho e^{it})}{f(z_0)}\right| \leq 1.$$

The continuous real-valued integrand is everywhere at most 1 and has average 1, so

$$\operatorname{Re}\left(\frac{f(z_0+\rho e^{it})}{f(z_0)}\right) = 1$$

for every t. Since

$$\left|\frac{f(z_0+\rho e^{it})}{f(z_0)}\right| \leq 1,$$

the imaginary part must be 0. Hence

$$\frac{f(z_0+\rho e^{it})}{f(z_0)} = 1$$

for every t, so $f = f(z_0)$ on the circle of radius ρ around z_0. Since ρ was arbitrary, $f = f(z_0)$ on $D(z_0, r)$. By Identity Theorem (5.3), f is constant on G, a contradiction. □

Exercise 5.3.1

Prove Fundamental Theorem of Algebra (4.17) using Maximum Modulus Principle (5.5).

Corollary 5.6 **Boundary Maximum Principle**

If $f : G \to \mathbb{C}$ is holomorphic, f is non-constant, and $K \subseteq G$ is compact and non-empty, then every point of K at which $|f|$ attains its maximum lies in ∂K. In particular,

$$\max_{z\in K}|f(z)| = \max_{z\in\partial K}|f(z)|.$$

Proof: Since K is compact and $|f|$ is continuous, $|f|$ attains a maximum on K, say at $z_0 \in K$.

If z_0 were in the interior of K, then there would be an $r > 0$ such that $D(z_0, r) \subseteq K$. Hence

$$|f(z)| \leq |f(z_0)|$$

for every $z \in D(z_0, r)$, so $|f|$ would have a local maximum at z_0. This contradicts Maximum Modulus Principle (5.5), since f is non-constant.

Therefore every point where $|f|$ achieves its maximum value on K lies in $K \setminus \operatorname{int}(K) = \partial K$. Thus the maximum over K is achieved on ∂K. □

Remark **Equivalence with the Maximum Modulus Principle**

The strengthened Boundary Maximum Principle above is equivalent to Maximum Modulus Principle (5.5). The implication from the Maximum Modulus Principle to the boundary principle is proved above. Conversely, suppose that the boundary principle holds and that $|f|$ has a local maximum at $z_0 \in G$. Choose $r > 0$ sufficiently small that

$$\overline{D}(z_0, r) \subseteq G$$

and

$$|f(z)| \leq |f(z_0)|$$

whenever $z \in \overline{D}(z_0, r)$. Then z_0 is an interior point of the compact set $\overline{D}(z_0, r)$ at which $|f|$ attains its maximum, contradicting the boundary principle. Thus $|f|$ has no local maximum in G.

Theorem 5.7 **Minimum Modulus Principle**

Suppose $f : G \to \mathbb{C}$ is holomorphic and nonconstant. If $|f|$ has a local minimum at $z_0 \in G$, then $f(z_0) = 0$.

Proof: If $f(z_0) \neq 0$, then f is nonzero on some disc $D(z_0, r) \subseteq G$. Hence $\frac{1}{f}$ is holomorphic on this disc. Since $|f|$ has a local minimum at z_0,

$$\left|\frac{1}{f}\right| = \frac{1}{|f|}$$

has a local maximum at z_0. By Maximum Modulus Principle (5.5), $\frac{1}{f}$ is constant on $D(z_0, r)$, so f is constant on $D(z_0, r)$. Identity Theorem (5.3) then forces f to be constant on G, a contradiction. Therefore $f(z_0) = 0$. □

Theorem 5.8 **Schwarz's Lemma**

Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, where $\mathbb{D}$ is the open unit disc, and suppose $f(0) = 0$. Then

$$|f(z)| \leq |z|$$

for $z \in \mathbb{D}$. Moreover, if $|f(z)| = |z|$ for some $0 \neq z \in \mathbb{D}$, then there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $f(z) = \lambda z$ for all $z \in \mathbb{D}$.

Proof: Since f is holomorphic and $f(0) = 0$, Holomorphic Functions Are Locally Analytic (4.12) gives a power series expansion near 0 of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Define

$$g(z) = \begin{cases} \frac{f(z)}{z}, \text{if } z \neq 0 \\ f'(0), \text{if } z = 0. \end{cases}$$

Dividing the power series by z gives, for $z \neq 0$ near 0,

$$\frac{f(z)}{z} = \sum_{n=1}^{\infty} a_n z^{n-1}.$$

The right-hand side is a power series whose value at $z = 0$ is $a_1 = f'(0)$, so it agrees with the definition of g at 0. Thus g is holomorphic at 0. On $\mathbb{D} \setminus \{0\}$, we have $g = \frac{f}{z}$, so g is holomorphic there as well. Hence g is holomorphic on $\mathbb{D}$.

Fix $0 < r < 1$. For $|z| = r$ we have

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r},$$

because $f(\mathbb{D}) \subseteq \mathbb{D}$. Since g is continuous on the compact disc $|z| \leq r$, $|g|$ attains a maximum there. If the maximum were attained at an interior point and were larger than the boundary maximum, then the Maximum Modulus Principle (5.5) would force g to be constant. Thus

$$|g(z)| \leq \frac{1}{r}$$

whenever $|z| \leq r$.

Now fix $z \in \mathbb{D}$. Letting $r \to 1$ with $|z| < r < 1$, we obtain $|g(z)| \leq 1$. Hence, for $z \neq 0$,

$$|f(z)| = |z||g(z)| \leq |z|,$$

and the inequality is also clear at $z = 0$.

Finally, suppose $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. Then $|g(z_0)| = 1$, so $|g|$ attains its maximum value at an interior point of $\mathbb{D}$. By Maximum Modulus Principle (5.5), g is constant on $\mathbb{D}$; say $g = \lambda$. Since $|g(z_0)| = 1$, we have $|\lambda| = 1$. Therefore $f(z) = \lambda z$ for every $z \in \mathbb{D}$. □

6 Laurent Theory and Isolated Singularities

6.1 Laurent Series and Convergence on Annuli

Note **Notation**

For $z_0 \in \mathbb{C}$ and $r > 0$, write

$$D(z_0, r) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

for the disc centered at z_0 with radius r, and write

$$D^*(z_0, r) := \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$$

for the corresponding punctured disc.

For $r > 0$, write

$$C_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| = r\}$$

for the circle centered at z_0 with radius r. When $C_r(z_0)$ is used as a contour, it is oriented counter-clockwise unless stated otherwise.

For $0 \leq r_1 < r_2 \leq \infty$, write

$$A(z_0, r_1, r_2) := \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$$

for the annulus centered at z_0 with inner radius r_1 and outer radius r_2. Thus $D^*(z_0, r) = A(z_0, 0, r)$.

Example

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

for $|z| < 1$. Moreover,

$$\begin{aligned}
\frac{1}{1-z} &= -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\
&= \sum_{n=-1}^{-\infty} -z^n
\end{aligned}$$

for $|z| > 1$.

Example

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Note that

$$\begin{aligned}\frac{1}{z-2} &= -\frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) \\ &= -\frac{1}{2}\sum_{n=0}^{\infty} 2^{-n} z^n\end{aligned}$$

for $|z| < 2$. Also,

$$-\frac{1}{z-1} = \sum_{n=-1}^{-\infty} -z^n$$

for $|z| > 1$. Therefore,

$$\frac{1}{(z-1)(z-2)} = \sum_{n=-1}^{-\infty} -z^n - \frac{1}{2}\sum_{n=0}^{\infty} z^{-n} z^n$$

for $1 < |z| < 2$.

Definition 6.1 — **Laurent Series**

Let $z_0 \in \mathbb{C}$ and let $(a_n)_{n\in\mathbb{Z}}$ be a bi-infinite sequence in $\mathbb{C}$ (equivalently, a function $\mathbb{Z} \to \mathbb{C}$). A *Laurent series* centered at z_0 with coefficients $(a_n)_{n\in\mathbb{Z}}$ is a formal power series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.$$

We say that the Laurent series converges at z if both

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{and} \quad \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

converge. In that case,

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n := \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}.$$

Note **Symmetric Partial Sums**

The *symmetric partial sums* of a Laurent series are

$$S_N(z) := \sum_{n=-N}^{N} a_n(z-z_0)^n.$$

If the Laurent series converges at z, then the symmetric partial sums converge, and

$$\lim_{N\to\infty} S_N(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}.$$

Theorem 6.1 **Annulus of Convergence for Laurent Series**

Let $z_0 \in \mathbb{C}$, and consider the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n.$$

Define

$$R_2 := \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

$$R_1 := \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}}$$

with the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$ in the definition of R_2. Then

1. The Laurent series converges absolutely and locally uniformly on the annulus
$$A(z_0, R_1, R_2).$$
2. The function
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$
is holomorphic on $A(z_0, R_1, R_2)$.

Proof: The non-negative part

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is an ordinary power series with radius of convergence R_2, by Cauchy-Hadamard Theorem (3.7).

For the negative part, put $w = \frac{1}{z-z_0}$. Then

$$\sum_{n=-1}^{-\infty} a_n (z - z_0)^n = \sum_{m=1}^{\infty} a_{-m} w^m.$$

This is a power series in w with radius of convergence $\frac{1}{R_1}$. Hence it converges whenever

$$|w| < \frac{1}{R_1},$$

which is equivalent to $|z - z_0| > R_1$.

Therefore both parts converge whenever $z \in A(z_0, R_1, R_2)$.

We now check local uniform convergence. Let $K \subseteq A(z_0, R_1, R_2)$ be compact. Since $z \mapsto |z - z_0|$ is continuous, the image

$$\{|z - z_0| : z \in K\}$$

is a compact subset of (R_1, R_2). Hence there exist numbers ρ_1, ρ_2 such that

$$R_1 < \rho_1 \leq |z - z_0| \leq \rho_2 < R_2$$

for every $z \in K$.

The positive part is a power series with radius of convergence R_2, so it converges uniformly on $|z - z_0| \leq \rho_2$. Hence it converges uniformly on K. For the negative part, the set K is sent by $w = \frac{1}{z - z_0}$ into

$$|w| \leq \frac{1}{\rho_1} < \frac{1}{R_1}.$$

Since the negative part is a power series in w with radius of convergence $\frac{1}{R_1}$, it converges uniformly on $|w| \leq \frac{1}{\rho_1}$, and therefore uniformly on K.

Thus both parts converge uniformly on every compact subset of $A(z_0, R_1, R_2)$, so the Laurent series converges locally uniformly there. Since each symmetric partial sum is holomorphic on $A(z_0, R_1, R_2)$, the Weierstrass Convergence Test (5.4) implies that the sum is holomorphic there. □

Theorem 6.2 — **Termwise Differentiation of Laurent Series**

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on the annulus $A(z_0, R_1, R_2)$. Then f may be differentiated term by term on this annulus, and

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1}.$$

Moreover, the differentiated Laurent series has the same annulus of convergence.

Proof: The non-negative part is an ordinary power series, so it may be differentiated term by term and its radius of convergence remains R_2.

For the negative part, write $w = \frac{1}{z-z_0}$. Then

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{m=1}^{\infty} a_{-m} w^m.$$

This is a power series in w, so it may be differentiated term by term in w and its radius of convergence remains $\frac{1}{R_1}$. Since $w = (z - z_0)^{-1}$ is holomorphic on the annulus, the chain rule gives

$$\frac{d}{dz}\left(a_{-m}(z - z_0)^{-m}\right) = -m a_{-m}(z - z_0)^{-m-1}.$$

Combining this with the differentiated non-negative part gives

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1}.$$

The convergence radii of the two differentiated pieces are unchanged, so the annulus of convergence is again $A(z_0, R_1, R_2)$. $\square$

Theorem 6.3 — **Cauchy's Theorem for the Annulus**

Let $0 \leq R_1 < R_2 \leq \infty$, and let f be holomorphic on $A(z_0, R_1, R_2)$. If $R_1 < r < s < R_2$, then

$$\int_{C_r(z_0)} f(z)\,\mathrm{d}z = \int_{C_s(z_0)} f(z)\,\mathrm{d}z.$$

Equivalently, the integral of f around the boundary of the closed annulus $\{z \in \mathbb{C} \mid r \leq |z - z_0| \leq s\}$ is zero, with the boundary orientation.

Proof: It suffices to prove the equivalent boundary statement. Fix $N \geq 3$ and set

$$\theta_k = 2\pi \frac{k}{N}, \quad p_k = z_0 + se^{i\theta_k}, \quad q_k = z_0 + re^{i\theta_k}$$

for $k = 0, ..., N$, where $p_N = p_0$ and $q_N = q_0$. Let P_s^N be the polygonal path through $p_0, p_1, ..., p_N$, oriented counter-clockwise, and let P_r^N be the polygonal path through $q_0, q_1, ..., q_N$, also oriented counter-clockwise.

For each $k = 0, ..., N - 1$, let T_k be the closed quadrilateral with vertices

$$q_k, p_k, p_{k+1}, q_{k+1}.$$

This quadrilateral lies in the annulus, and splitting it along a diagonal expresses it as the union of two triangles. Since f is holomorphic on a neighbourhood of each triangle, Cauchy's Theorem for Triangles (Goursat's Lemma) (4.7) gives

$$\int_{\partial T_k} f(z) \,\mathrm{d}z = 0$$

with positive boundary orientation.

Summing over k, the integrals over the radial cuts cancel in pairs, because each cut is traversed once in each direction. The remaining outer boundary is P_s^N, while the remaining inner boundary is $-P_r^N$. Therefore

$$\int_{P_s^N} f(z) \,\mathrm{d}z - \int_{P_r^N} f(z) \,\mathrm{d}z = 0,$$

so

$$\int_{P_s^N} f(z) \,\mathrm{d}z = \int_{P_r^N} f(z) \,\mathrm{d}z.$$

As $N \to \infty$, the polygonal paths P_s^N and P_r^N converge uniformly to the corresponding circles $C_s(z_0)$ and $C_r(z_0)$, and their lengths remain bounded. Since f is uniformly continuous on the compact annulus $\{z \in \mathbb{C} \mid r \leq |z - z_0| \leq s\}$, the polygonal integrals converge to the circular integrals. Passing to the limit gives

$$\int_{C_s(z_0)} f(z) \,\mathrm{d}z = \int_{C_r(z_0)} f(z) \,\mathrm{d}z,$$

as required. □

Proof: Alternative proof. For $R_1 < r < R_2$, let

$$I(r) = \int_{C_r(z_0)} f(z) \,\mathrm{d}z.$$

Using the parametrization

$$t \mapsto z_0 + re^{it}, \quad 0 \le t \le 2\pi,$$

of $C_r(z_0)$, we can write

$$I(r) = \int_0^{2\pi} f(z_0 + re^{it}) ire^{it} \, \mathrm{d}t.$$

The integrand in the preceding integral, as a function of the two real variables r and t, has continuous partial derivatives. Thus we may differentiate under the integral sign:

$$\frac{\mathrm{d}I(r)}{\mathrm{d}r} = \int_0^{2\pi} \partial_r [f(z_0 + re^{it}) ire^{it}] \, \mathrm{d}t.$$

We have

$$\begin{aligned} \partial_r [f(z_0 + re^{it}) ire^{it}] &= f'(z_0 + re^{it}) ire^{2it} + f(z_0 + re^{it}) ie^{it} \\ &= \partial_t [f(z_0 + re^{it}) e^{it}]. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\mathrm{d}I(r)}{\mathrm{d}r} &= \int_0^{2\pi} \partial_t [f(z_0 + re^{it}) e^{it}] \, \mathrm{d}t \\ &= [f(z_0 + re^{it}) e^{it}]_{t=0}^{t=2\pi} \\ &= 0. \end{aligned}$$

Hence I is constant on (R_1, R_2), and therefore

$$\int_{C_r(z_0)} f(z) \, \mathrm{d}z = \int_{C_s(z_0)} f(z) \, \mathrm{d}z$$

whenever $R_1 < r < s < R_2$. □

6.2 Cauchy Formulas and Laurent Expansions on Annuli

Theorem 6.4 **Cauchy's Formula for an Annulus**

Let $0 \le R_1 < R_2 \le \infty$, let $z_0 \in \mathbb{C}$, and put $\Omega = A(z_0, R_1, R_2)$. Suppose that f is holomorphic on Ω. For $R_1 < r < R_2$, let $C_r(z_0)$ be as in the notation above. If $w \in \Omega$ and

$$R_1 < r_1 < |w - z_0| < r_2 < R_2,$$

then

$$f(w) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(z)}{z - w} \, \mathrm{d}z - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(z)}{z - w} \, \mathrm{d}z.$$

Proof: Fix w, r_1, and r_2 as in the statement. Define $g : \Omega \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w}, & z \neq w \\ f'(w), & z = w. \end{cases}$$

Then g is holomorphic on Ω: it is holomorphic on $\Omega \setminus \{w\}$, and the value $g(w) = f'(w)$ removes the apparent singularity at w.

By Cauchy's Theorem for the Annulus (6.3) applied to g on Ω,

$$\int_{C_{r_1}(z_0)} g(z)\,\mathrm{d}z = \int_{C_{r_2}(z_0)} g(z)\,\mathrm{d}z.$$

Expanding g and moving terms gives

$$\int_{C_{r_2}(z_0)} \frac{f(z)}{z-w}\,\mathrm{d}z - \int_{C_{r_1}(z_0)} \frac{f(z)}{z-w}\,\mathrm{d}z = f(w)\int_{C_{r_2}(z_0)} \frac{1}{z-w}\,\mathrm{d}z - f(w)\int_{C_{r_1}(z_0)} \frac{1}{z-w}\,\mathrm{d}z.$$

Since $|w-z_0| < r_2$, the point w lies inside $C_{r_2}(z_0)$. Hence Cauchy's Formula for the Circle (4.9), applied to the constant function 1, gives

$$\int_{C_{r_2}(z_0)} \frac{1}{z-w}\,\mathrm{d}z = 2\pi i.$$

Since $|w-z_0| > r_1$, the function $z \mapsto \frac{1}{z-w}$ is holomorphic on a neighbourhood of the closed disc $\{z \in \mathbb{C} \mid |z-z_0| \leq r_1\}$. By Cauchy's Theorem for Convex Regions (4.8),

$$\int_{C_{r_1}(z_0)} \frac{1}{z-w}\,\mathrm{d}z = 0.$$

Therefore

$$\int_{C_{r_2}(z_0)} \frac{f(z)}{z-w}\,\mathrm{d}z - \int_{C_{r_1}(z_0)} \frac{f(z)}{z-w}\,\mathrm{d}z = 2\pi i f(w),$$

and division by $2\pi i$ gives the formula. □

Theorem 6.5 **Laurent Expansion Theorem**

Let $0 \leq R_1 < R_2 \leq \infty$, let $z_0 \in \mathbb{C}$, and put $\Omega = A(z_0, R_1, R_2)$. If f is holomorphic on Ω, then f has a Laurent series representation on Ω:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad z \in \Omega,$$

where, for any r satisfying $R_1 < r < R_2$,

$$a_n = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta, \quad n \in \mathbb{Z}.$$

Proof: First note that the coefficient formula is independent of the admissible radius r. Indeed, for fixed $n \in \mathbb{Z}$, the function

$$\zeta \mapsto \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$$

is holomorphic on Ω. Hence Cauchy's Theorem for the Annulus (6.3) shows that its integrals over any two centered circles lying in Ω are equal.

Now fix $z \in \Omega$. Choose radii r_1, r_2 such that

$$R_1 < r_1 < |z - z_0| < r_2 < R_2.$$

By Cauchy's Formula for an Annulus (6.4),

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

On $C_{r_2}(z_0)$ we have $|z - z_0| < |\zeta - z_0| = r_2$, so

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}, \end{aligned}$$

uniformly for $\zeta \in C_{r_2}(z_0)$. Therefore

$$\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta \right) (z - z_0)^n.$$

On $C_{r_1}(z_0)$ we have $|\zeta - z_0| = r_1 < |z - z_0|$, so

$$\begin{aligned} \frac{1}{\zeta - z} &= -\frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} \\ &= -\sum_{m=0}^{\infty} \frac{(\zeta - z_0)^m}{(z - z_0)^{m+1}}, \end{aligned}$$

uniformly for $\zeta \in C_{r_1}(z_0)$. Hence

$$-\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \sum_{m=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} f(\zeta)(\zeta - z_0)^m \, \mathrm{d}\zeta \right) (z - z_0)^{-m-1}.$$

Re-indexing the last sum by $n = -m - 1$ gives precisely the negative-power part of the Laurent series, because

$$(\zeta - z_0)^m = (\zeta - z_0)^{-n-1}.$$

Combining the two expansions gives

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Since the coefficients are independent of the chosen radii r_1 and r_2, this representation holds for every $z \in \Omega$. The same argument, with r_1, r_2 chosen uniformly around a compact subannulus, also gives locally uniform convergence on Ω. □

6.3 Classification of Isolated Singularities

Definition 6.2

Suppose $f : G \to \mathbb{C}$ is a holomorphic function. A point z_0 is an *isolated singularity* if $z_0 \notin G$, but there exists $R > 0$ such that $D^*(z_0, R) \subseteq G$.

Note **Isolated Singularity**

If z_0 is an isolated singularity, then f is holomorphic on some punctured disc $D^*(z_0, R) = A(z_0, 0, R)$. Since this is an annulus, Laurent Expansion Theorem (6.5) gives a Laurent series representation for f there.

Example

$f(z) = z$ on $G = \{z \in \mathbb{C} \mid z \neq 0\}$.

Definition 6.3 **Removable Singularity**

Let $f : G \to \mathbb{C}$ be holomorphic. We say that $z_0 \notin G$ is a removable singularity if there exists a holomorphic extension of f on $G \cup \{z_0\}$.

Example

Let $f(z) = \frac{z}{e^z - 1}$ on $G = \{z \in \mathbb{C} \mid z \neq 0\}$. Since

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} z^n,$$

we have, for $z \neq 0$,

$$f(z) = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n!} z^{n-1}}.$$

The denominator is holomorphic near 0 and has value 1 at 0, so this formula defines a holomorphic extension across 0. Therefore 0 is a removable singularity.

Definition 6.4

An isolated singularity is a *pole of order* m if in the Laurent series representation near z_0, we have

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$$

with $a_{-m} \neq 0$.

Order 1 poles are *simple poles.*

Example

$f(z) = \frac{1}{z}$ is a simple pole.

Proposition 6.6 **Characterization of Poles**

Let $f : G \to \mathbb{C}$ be holomorphic, and let z_0 be an isolated singularity of f. Then z_0 is a pole of order m if and only if there is a holomorphic function g near z_0 such that

$$f(z) = (z - z_0)^{-m} g(z)$$

for $z \neq z_0$ near z_0, with $g(z_0) \neq 0$.

Proof: Suppose z_0 is a pole of order m. Then the Laurent series of f near z_0 has the form

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n,$$

with $a_{-m} \neq 0$. Hence

$$f(z) = (z - z_0)^{-m} \sum_{n=0}^{\infty} a_{n-m} (z - z_0)^n$$

The power series

$$g(z) = \sum_{n=0}^{\infty} a_{n-m} (z - z_0)^n$$

is holomorphic near z_0, and $g(z_0) = a_{-m} \neq 0$.

Conversely, if

$$f(z) = (z - z_0)^{-m} g(z)$$

with g holomorphic near z_0 and $g(z_0) \neq 0$, then the Taylor series of g at z_0 has nonzero constant term. Multiplying by $(z - z_0)^{-m}$ gives a Laurent series whose first nonzero negative-power term is $(z - z_0)^{-m}$. Thus z_0 is a pole of order m. □

Definition 6.5 — **Essential Singularity**

An isolated singularity z_0 is an *essential singularity* if the Laurent series of f near z_0 has infinitely many nonzero negative-power terms.

Example

The function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at 0, since

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

has infinitely many nonzero negative-power terms.

Definition 6.6 — **∞ as an Isolated Singularity**

Let $f : G \to \mathbb{C}$ be holomorphic. We say that ∞ is an isolated singularity for f if there exists $R > 0$ such that

$$\{z \in \mathbb{C} \mid |z| > R\} \subseteq G.$$

Definition 6.7 — **Type of Singularity at ∞**

Suppose ∞ is an isolated singularity for f. Define

$$g(w) = f\left(\frac{1}{w}\right)$$

for $w \in D^*\left(0, \frac{1}{R}\right)$. We say that ∞ is a removable singularity, a pole of order m, or an essential singularity for f if 0 has the corresponding type of isolated singularity for g.

Example

The function $f(z) = z$ has a simple pole at ∞, since

$$f\left(\frac{1}{w}\right) = \frac{1}{w}$$

has a simple pole at 0.

Example

The function $f(z) = \frac{1}{z}$ has a removable singularity at ∞, since

$$f\left(\frac{1}{w}\right) = w$$

has a removable singularity at 0.

6.4 Removable Singularities, Poles, and Essential Singularities

Note

We fix a holomorphic function $f : G \to \mathbb{C}$, where G is an open set.

Theorem 6.7 **Removable Singularity Theorem**

Let z_0 be an isolated singularity of $f : G \to \mathbb{C}$. Then z_0 is removable if and only if f is bounded on some punctured disc around z_0. That is, there exist $r > 0$ and $M > 0$ such that $D^*(z_0, r) \subseteq G$ and $|f(z)| \leq M$ whenever $z \in D^*(z_0, r)$.

Proof: First suppose z_0 is removable. Let $F : G \cup \{z_0\} \to \mathbb{C}$ be a holomorphic extension of f. Since F is continuous at z_0, there is some $r > 0$ such that F is bounded on compact set $\overline{D}(z_0, r)$. Hence f is bounded on $D^*(z_0, r)$.

Conversely, suppose $|f(z)| \leq M$ whenever $z \in D^*(z_0, r)$. Write the Laurent series of f near z_0 as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We show that all negative coefficients vanish. Fix $n \geq 1$. For $0 < \rho < r$, by Laurent Expansion Theorem (6.5) applied on $C_\rho(z_0)$,

$$a_{-n} = \frac{1}{2\pi i} \int_{C_\rho(z_0)} f(z)(z - z_0)^{n-1} \, \mathrm{d}z.$$

Therefore, by ML Inequality (4.5),

$$|a_{-n}| \leq \frac{1}{2\pi} \cdot M\rho^{n-1} \cdot 2\pi\rho = M\rho^n.$$

Since this holds for every $0 < \rho < r$, letting $\rho \to 0$ gives $a_{-n} = 0$. Thus $a_k = 0$ for all $k < 0$, so the Laurent series defines a holomorphic extension across z_0. Therefore z_0 is removable. □

Theorem 6.8 **Pole Characterization by Growth**

Suppose z_0 is an isolated singularity of f. Then z_0 is a pole if and only if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Proof: First suppose z_0 is a pole of order m. By Characterization of Poles (6.6), the function

$$g(z) = (z - z_0)^m f(z)$$

has a removable singularity at z_0, and its removable extension satisfies $g(z_0) \neq 0$. Hence there are $r > 0$ and $c > 0$ such that $|g(z)| \geq c$ whenever $z \in D(z_0, r)$. Thus, for $z \in D^*(z_0, r)$,

$$|f(z)| = \frac{|g(z)|}{|z - z_0|^m} \geq \frac{c}{|z - z_0|^m}.$$

Letting $z \to z_0$ gives $|f(z)| \to \infty$.

Conversely, suppose

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Then there is $r > 0$ such that $f(z) \neq 0$ whenever $z \in D^*(z_0, r)$. Define $h(z) = \frac{1}{f(z)}$ on $D^*(z_0, r)$. Since $|h(z)| \to 0$ as $z \to z_0$, the function h is bounded near z_0, so by Removable Singularity Theorem (6.7), h has a removable singularity at z_0. Define its extension by $h(z_0) = 0$.

The zero of h at z_0 has finite order. Indeed, h is not identically zero on any punctured disc around z_0, since f is finite there, so h cannot have a zero of infinite order at z_0. Thus

$$h(z) = (z - z_0)^m k(z)$$

for some $m \geq 1$, where k is holomorphic near z_0 and $k(z_0) \neq 0$. Therefore

$$f(z) = \frac{1}{h(z)} = (z - z_0)^{-m} \frac{1}{k(z)},$$

where $\frac{1}{k}$ is holomorphic near z_0. By Characterization of Poles (6.6), z_0 is a pole of order m.
□

Theorem 6.9 **Great Picard's Theorem**

Let z_0 be an essential singularity of $f : G \to \mathbb{C}$. For every $\varepsilon > 0$ such that $D^*(z_0, \varepsilon) \subseteq G$, the image $f(D^*(z_0, \varepsilon))$ omits at most one complex number.

Note

The proof of Picard's theorem is out of scope.

Theorem 6.10 **Casorati-Weierstrass Theorem**

Let z_0 be an essential singularity of $f : G \to \mathbb{C}$. For every $\varepsilon > 0$ such that $D^*(z_0, \varepsilon) \subseteq G$, the image $f(D^*(z_0, \varepsilon))$ is dense in $\mathbb{C}$.

Proof: Suppose not. Then there exist $\varepsilon > 0$, $w \in \mathbb{C}$, and $\delta > 0$ such that

$$|f(z) - w| \geq \delta$$

whenever $z \in D^*(z_0, \varepsilon)$. In particular, $f(z) \neq w$ on this punctured disc, so

$$g(z) = \frac{1}{f(z) - w}$$

is holomorphic there. Moreover, $|g(z)| \leq \frac{1}{\delta}$, so by Removable Singularity Theorem (6.7), g has a removable singularity at z_0.

Let G_0 be the holomorphic extension of g to a disc centered at z_0. We have two cases:

1. If $G_0(z_0) \neq 0$, then $\frac{1}{G_0}$ is holomorphic in a smaller disc centered at z_0, so

$$f(z) = w + \frac{1}{G_0}(z)$$

   has a removable singularity at z_0, a contradiction.
2. If $G_0(z_0) = 0$, then $|g(z)| \to 0$ as $z \to z_0$, so

$$|f(z) - w| = \frac{1}{|g(z)|} \to \infty.$$

   By Pole Characterization by Growth (6.8), $f - w$ has a pole at z_0, and hence f has a pole at z_0, again a contradiction.

Therefore the image of every punctured disc around z_0 is dense in $\mathbb{C}$. □

7 Winding Numbers, Global Cauchy Theory, and Residues

Note **Goal**

We want to extend Cauchy's Theorem for Convex Regions (4.8) to simply connected regions. Convexity was useful because every triangle and line segment stayed inside the region. For simply connected regions, we need a topological replacement for that condition. The next results set this up using continuous logarithms along curves, increments of argument, and winding numbers.

7.1 Continuous Logarithms and Exponential Lifting Along Curves

Lemma 7.1 **Disc Partition Lemma for Nonvanishing Curves**

Let $a < b$, and let $\varphi : [a, b] \to \mathbb{C}^*$ be continuous. Then there are a partition

$$a = t_0 < t_1 < \ldots < t_n = b$$

and open discs $D_0, \ldots, D_{n-1}$, none of which contains 0, such that

$$\varphi([t_j, t_{j+1}]) \subseteq D_j$$

for every $j = 0, \ldots, n - 1$.

Proof: Since φ is continuous on the compact interval $[a, b]$, it is uniformly continuous. Moreover, $\varphi(t) \neq 0$ for every $t \in [a, b]$, so the Extreme Value Theorem gives

$$m := \min_{t \in [a,b]} |\varphi(t)| > 0.$$

By uniform continuity, there exists $\delta > 0$ such that

$$|s - t| < \delta \quad \Longrightarrow \quad |\varphi(s) - \varphi(t)| < \frac{m}{2}.$$

Choose a partition whose mesh is smaller than δ, and set

$$D_j = D\left(\varphi(t_j), \frac{m}{2}\right).$$

If $t \in [t_j, t_{j+1}]$, then

$$|\varphi(t) - \varphi(t_j)| < \frac{m}{2},$$

so $\varphi([t_j, t_{j+1}]) \subseteq D_j$. Finally, $|\varphi(t_j)| \geq m > \frac{m}{2}$, so $0 \notin D_j$. □

Theorem 7.2 **Path Lifting for the Exponential Map**

Let $a < b$, and let $\varphi : [a, b] \to \mathbb{C}^*$ be continuous. Then there exists a continuous map $\psi : [a, b] \to \mathbb{C}$ such that

$$e^{\psi(t)} = \varphi(t)$$

for every $t \in [a, b]$. Moreover, ψ is unique up to adding a constant in $2\pi i\mathbb{Z}$.

Proof: By Disc Partition Lemma for Nonvanishing Curves (7.1), choose a partition

$$a = t_0 < t_1 < \ldots < t_n = b$$

and discs $D_0, \ldots, D_{n-1}$ such that $\varphi([t_j, t_{j+1}]) \subseteq D_j$ and $0 \notin D_j$ for every j. For each j, choose a branch L_j of log on D_j, and define

$$\psi_j = L_j \circ \varphi$$

on $[t_j, t_{j+1}]$. Then ψ_j is continuous and

$$e^{\psi_j(t)} = \varphi(t).$$

We now adjust these local lifts inductively. Set $\tilde{\psi}_0 = \psi_0$. Suppose $\tilde{\psi}_{j-1}$ has been defined on $[t_{j-1}, t_j]$. At the common endpoint t_j,

$$e^{\tilde{\psi}_{j-1}(t_j)} = \varphi(t_j) = e^{\psi_j(t_j)}.$$

Therefore

$$c_j := \tilde{\psi}_{j-1}(t_j) - \psi_j(t_j) \in 2\pi i\mathbb{Z}.$$

Define

$$\tilde{\psi}_j(t) := \psi_j(t) + c_j$$

for $t \in [t_j, t_{j+1}]$. Then

$$\tilde{\psi}_j(t_j) = \tilde{\psi}_{j-1}(t_j)$$

and, since $e^{c_j} = 1$,

$$e^{\tilde{\psi}_j(t)} = e^{\psi_j(t)} = \varphi(t).$$

Continuing for $j = 1, ..., n-1$ gives local lifts that agree at every partition point. Define ψ by setting $\psi(t) = \tilde{\psi}_j(t)$ on each $[t_j, t_{j+1}]$. The finite pasting lemma shows that ψ is continuous on $[a, b]$, and the preceding identity gives $e^{\psi(t)} = \varphi(t)$ everywhere.

If ψ and η are two such choices, then $\psi - \eta$ is a continuous map from $[a, b]$ to the discrete set $2\pi i\mathbb{Z}$. Since $[a, b]$ is connected, $\psi - \eta$ is constant. □

Proposition 7.3 **Integral Construction of an Exponential Lift**

Let $\varphi : [a, b] \to \mathbb{C}^*$ be piecewise-C^1. Fix $c \in \mathbb{C}$ such that $e^c = \varphi(a)$, and set

$$\psi(t) = c + \int_a^t \frac{\varphi'(s)}{\varphi(s)} \, \mathrm{d}s.$$

Then ψ is continuous, piecewise-C^1, and $\psi'(t) = \frac{\varphi'(t)}{\varphi(t)}$ wherever φ' exists. Hence $\left(\varphi e^{-\psi}\right)' = 0$ on each smooth piece, and since $\varphi(a)e^{-\psi(a)} = 1$, we get

$$e^{\psi(t)} = \varphi(t)$$

for every $t \in [a, b]$. Thus ψ is a continuous logarithm of φ, and its increment is

$$\psi(b) - \psi(a) = \int_a^b \frac{\varphi'(t)}{\varphi(t)} \, \mathrm{d}t.$$

Corollary 7.4 **Exponential Lifting of a Pullback Along a Curve**

Let $\gamma : [a, b] \to \mathbb{C}$ be continuous, and let f be continuous on $R(\gamma)$. If

$$(f \circ \gamma)(t) \neq 0$$

for every $t \in [a, b]$, then there exists a continuous map $\psi : [a, b] \to \mathbb{C}$ such that

$$e^{\psi(t)} = (f \circ \gamma)(t)$$

for every $t \in [a, b]$.

Proof: Let $\varphi = f \circ \gamma$. Since γ is continuous and f is continuous on $R(\gamma)$, the map $\varphi : [a, b] \to \mathbb{C}$ is continuous. By assumption,

$$\varphi(t) = (f \circ \gamma)(t) \neq 0$$

for every $t \in [a, b]$, so φ is a continuous map from $[a, b]$ into $\mathbb{C}^*$.

By Path Lifting for the Exponential Map (7.2), there exists a continuous map $\psi : [a, b] \to \mathbb{C}$ such that $e^{\psi(t)} = \varphi(t)$ for every $t \in [a, b]$. Since $\varphi = f \circ \gamma$, this gives

$$e^{\psi(t)} = (f \circ \gamma)(t)$$

for every $t \in [a, b]$. □

7.2 Increments of Logarithm and Argument; Winding Numbers

Definition 7.1 **Increment in Log**

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous curve, and let f be continuous and nonzero on $R(\gamma)$. Choose a continuous map $\psi : [a, b] \to \mathbb{C}$ such that

$$e^{\psi(t)} = (f \circ \gamma)(t)$$

for every $t \in [a, b]$. The *increment in log* of f along γ is

$$\Delta(\log f, \gamma) = \psi(b) - \psi(a).$$

Note

Increment in Log is well-defined: if $\tilde{\psi} : [a, b] \to \mathbb{C}$ is another continuous map with $e^{\tilde{\psi}(t)} = (f \circ \gamma)(t)$ for every $t \in [a, b]$, then there exists $m \in \mathbb{Z}$ such that

$$\tilde{\psi}(t) = \psi(t) + 2\pi i m$$

for every $t \in [a, b]$. Hence

$$\tilde{\psi}(b) - \tilde{\psi}(a) = (\psi(b) + 2\pi i m) - (\psi(a) + 2\pi i m) = \psi(b) - \psi(a).$$

Example

Let $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$, and let $f(z) = z$. We can take $\psi(t) = it$, so

$$\Delta(\log z, \gamma) = \psi(2\pi) - \psi(0) = 2\pi i.$$

Definition 7.2 **Increment in Argument**

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous curve, and let f be continuous and nonzero on $R(\gamma)$. The *increment in argument* of f along γ is

$$\Delta(\arg f, \gamma) := \operatorname{Im}(\Delta(\log f, \gamma)) \in \mathbb{R}.$$

Definition 7.3 **Winding Number**

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous closed curve, and let $z_0 \notin R(\gamma)$. The *winding number* of γ about z_0 is

$$\operatorname{Ind}_{z=z_0}(\gamma) = \frac{1}{2\pi}\Delta(\arg(z - z_0), \gamma).$$

This is an integer.

Example

Let $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$. For $z_0 = 0$, we have

$$\Delta(\arg(z - 0), \gamma) = \Delta(\arg z, \gamma) = 2\pi.$$

Hence

$$\operatorname{Ind}_{z=0}(\gamma) = \frac{1}{2\pi} \cdot 2\pi = 1.$$

Note

If γ is closed and f is continuous and nonzero on $R(\gamma)$, then $(f \circ \gamma)(a) = (f \circ \gamma)(b)$. Thus the starting and ending values have the same magnitude, so the real part of $\Delta(\log f, \gamma)$ is 0. The only possible net change is in the continuously chosen argument, which can change by an integer multiple of 2π. Hence $\Delta(\log f, \gamma)$ is an integer multiple of $2\pi i$, and

$$\Delta(\arg f, \gamma) = \frac{1}{i}\Delta(\log f, \gamma)$$

is an integer multiple of 2π.

Proposition 7.5 **Integral Formula for the Winding Number**

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise-C^1 closed curve, and let $z_0 \notin R(\gamma)$. Set $\varphi(t) = \gamma(t) - z_0$. By Integral Construction of an Exponential Lift (7.3), we may choose ψ so that $e^{\psi(t)} = \varphi(t)$ and

$$\psi(b) - \psi(a) = \int_a^b \frac{\varphi'(t)}{\varphi(t)}\, dt = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt.$$

Therefore $\Delta(\log(z - z_0), \gamma) = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt$. Therefore

$$\operatorname{Ind}_{z=z_0}(\gamma) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt.$$

7.3 Topological Properties of the Winding Number

Proposition 7.6 **Continuity of the Winding Number**

Let γ be a closed piecewise-C^1 curve. Then $z_0 \mapsto \operatorname{Ind}_{z=z_0}(\gamma)$ is continuous on $\mathbb{C} \setminus R(\gamma)$.

Proof: Fix $z_0 \in \mathbb{C} \setminus R(\gamma)$. Since $R(\gamma)$ is compact, it is closed, so $\mathbb{C} \setminus R(\gamma)$ is open. Thus there is $d > 0$ with $B(z_0, d) \subseteq \mathbb{C} \setminus R(\gamma)$.

If $|z_1 - z_0| < \frac{d}{2}$, then $z_1 \notin R(\gamma)$ and $|z - z_1| \geq \frac{d}{2}$ for every $z \in R(\gamma)$. Hence

$$\left|\frac{1}{z - z_1} - \frac{1}{z - z_0}\right| = \frac{|z_1 - z_0|}{|z - z_1||z - z_0|} \leq \frac{2|z_1 - z_0|}{d^2}.$$

Using Integral Formula for the Winding Number (7.5) and ML Inequality (4.5),

$$\left|\operatorname{Ind}_{z=z_1}(\gamma) - \operatorname{Ind}_{z=z_0}(\gamma)\right| \leq \frac{L(\gamma)}{2\pi} \cdot \frac{2|z_1 - z_0|}{d^2}.$$

The right-hand side tends to 0 as $z_1 \to z_0$, so $\operatorname{Ind}_{z=z_0}(\gamma)$ is continuous on $\mathbb{C} \setminus R(\gamma)$. □

Corollary 7.7 **Local Constancy of the Winding Number**

Let γ be a closed piecewise-C^1 curve. Then the function $z_0 \mapsto \operatorname{Ind}_{z=z_0}(\gamma)$ is constant on each connected component of $\mathbb{C} \setminus R(\gamma)$.

Proof: By Continuity of the Winding Number (7.6), $z_0 \mapsto \operatorname{Ind}_{z=z_0}(\gamma)$ is continuous on each connected component of $\mathbb{C} \setminus R(\gamma)$. It also takes values in $\mathbb{Z}$. The image of a connected set under a continuous map is connected, and the only connected subsets of $\mathbb{Z}$ are singletons. Therefore the winding number is constant on each connected component. □

Proposition 7.8 **Vanishing on the Unbounded Component**

Let γ be a closed piecewise-C^1 curve. Then $\operatorname{Ind}_{z=z_0}(\gamma) = 0$ for every z_0 in the unbounded connected component of $\mathbb{C} \setminus R(\gamma)$.

Proof: Since $R(\gamma)$ is compact, it is bounded. Choose $M > 0$ such that $|z| \leq M$ for every $z \in R(\gamma)$. If $|z_0| > M + \frac{\ell(\gamma)}{2\pi}$, then $z_0 \notin R(\gamma)$ and

$$\left|\operatorname{Ind}_{z=z_0}(\gamma)\right| = \left|\frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z\right| \leq \frac{\ell(\gamma)}{2\pi(|z_0| - M)} < 1.$$

But $\operatorname{Ind}_{z=z_0}(\gamma) \in \mathbb{Z}$, so $\operatorname{Ind}_{z=z_0}(\gamma) = 0$ for such a point z_0.

This point lies in the unbounded connected component of $\mathbb{C} \setminus R(\gamma)$. By Local Constancy of the Winding Number (7.7), the winding number is constant on that component, so it is 0 everywhere on the unbounded component. □

Theorem 7.9 **Jordan Curve Theorem**

If $\gamma : [a, b] \to \mathbb{C}$ is continuous, $\gamma(a) = \gamma(b)$, and γ is injective on $[a, b)$, then $\mathbb{C} \setminus R(\gamma)$ has exactly two connected components.

7.4 Contours and the Separation Lemma

Definition 7.4

A *contour* is a possibly empty finite formal sum

$$\Gamma = \sum_{j=1}^{m} n_j \gamma_j,$$

where $n_j \in \mathbb{Z}$ and each γ_j is a piecewise-C^1 closed curve. Its range is defined to be

$$R(\Gamma) = \cup_{j=1}^{m} R(\gamma_j).$$

Definition 7.5

If $\Gamma = \sum_{j=1}^{m} n_j \gamma_j$ is a contour and f is continuous on $R(\Gamma)$, then

$$\int_{\Gamma} f(z) \, \mathrm{d}z = \sum_{j=1}^{m} n_j \int_{\gamma_j} f(z) \, \mathrm{d}z.$$

Definition 7.6 **Winding Number**

If

$$\Gamma = \sum_{j=1}^{m} n_j \gamma_j$$

is a contour and $z_0 \notin R(\Gamma)$, then

$$\operatorname{Ind}_{z=z_0}(\Gamma) := \sum_{j=1}^{m} n_j \operatorname{Ind}_{z=z_0}(\gamma_j).$$

Definition 7.7

1. A contour Γ is *simple* if $\operatorname{Ind}_{z=z_0}(\Gamma) \in \{0, 1\}$ for every $z_0 \notin R(\Gamma)$.
2. The *inside* of Γ is

$$\operatorname{int}(\Gamma) := \left\{ z_0 \in \mathbb{C} \setminus R(\Gamma) \mid \operatorname{Ind}_{z=z_0}(\Gamma) = 1 \right\}.$$

Theorem 7.10 **Separation Lemma**

Suppose $G \subseteq \mathbb{C}$ is open and $K \subseteq G$ is compact. Then there exists a simple contour Γ such that

$$R(\Gamma) \subseteq G \setminus K \quad \text{and} \quad K \subseteq \text{int}(\Gamma).$$

Proof: If $K = \varnothing$, take the empty contour. Now suppose $K \neq \varnothing$. Since K is compact and $\mathbb{C} \setminus G$ is closed, and since $K \subseteq G$, there exists $\delta > 0$ such that

$$|z - w| \geq \delta$$

whenever $z \in K$ and $w \in \mathbb{C} \setminus G$.

Consider the square grid in $\mathbb{C}$ with side length $\eta = \frac{\delta}{3}$. Let $S_1, S_2, \ldots, S_p$ be the closed grid squares that intersect K. There are only finitely many such squares because K is bounded. If $z \in S_j$, choose $w \in S_j \cap K$. Then

$$|z - w| \leq \sqrt{2}\eta < \delta,$$

so $z \in G$. Hence

$$U = \cup_{j=1}^{p} S_j \subseteq G.$$

Regard the selected squares as an integral 2-chain

$$C = \sum_{j=1}^{p} S_j.$$

Its boundary ∂C is the formal sum of the counter-clockwise boundaries of the selected squares. Every edge shared by two selected squares appears in ∂C twice with opposite orientations, so those interior edges cancel. Thus ∂C consists exactly of the remaining boundary edges of U.

The chain ∂C is a finite union of closed polygonal curves. Let Γ be the contour obtained by taking these boundary curves with their induced orientations. Then $R(\Gamma)$ is exactly the boundary of U.

We claim that $K \cap R(\Gamma) = \varnothing$. Indeed, if $z \in K$ lies on a grid edge, then every grid square adjacent to that edge also intersects K, so that edge is shared by selected squares and cancels. The same argument applies at grid vertices. Therefore every point of K lies in the interior of U, and no point of K lies on $R(\Gamma)$. Since $U \subseteq G$, this gives $R(\Gamma) \subseteq G \setminus K$.

Finally, the winding number of the positively oriented boundary of a square is 1 inside the square and 0 outside the square. By additivity of winding number over contours, the cancellations above imply that $\text{Ind}_{z=z_0}(\Gamma)$ is 1 on the interior of U and 0 outside U for every $z_0 \notin R(\Gamma)$. Hence Γ is simple, and since K lies in the interior of U, we have $K \subseteq \text{int}(\Gamma)$.

Note: See relation to 1-chain, homology. □

7.5 Cauchy's Theorem and Formula for General Contours

Corollary 7.11 **Cauchy's Formula for a Separating Contour**

If we have Γ as constructed in Separation Lemma (7.10), and if $f : G \to \mathbb{C}$ is holomorphic, then for $z_0 \in K$,

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

Proof: In the proof of Separation Lemma (7.10), Γ is the boundary of a finite union of closed squares contained in G. Equivalently, Γ is obtained from the formal sum of the positively oriented boundaries of those squares after cancelling shared edges.

Let h be holomorphic on G. By Cauchy's Theorem for Convex Regions (4.8), $\int_{\partial S_j} h(z) \, \mathrm{d}z = 0$ for each selected square S_j. Summing over the selected squares and using the edge cancellations gives

$$\int_\Gamma h(z) \, \mathrm{d}z = 0.$$

Fix $z_0 \in K$ and define

$$h(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ f'(z_0), & z = z_0 \end{cases}$$

The singularity at z_0 is removable, so h is holomorphic on G. Therefore

$$0 = \int_\Gamma h(z) \, \mathrm{d}z = \int_\Gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z - f(z_0) \int_\Gamma \frac{1}{z - z_0} \, \mathrm{d}z.$$

Since $K \subseteq \mathrm{int}(\Gamma)$, we have $\mathrm{Ind}_{z=z_0}(\Gamma) = 1$, and hence

$$\int_\Gamma \frac{1}{z - z_0} \, \mathrm{d}z = 2\pi i.$$

Thus

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

□

Theorem 7.12 **Generalized Cauchy Theorem**

Let G be an open set, let Γ be a contour with $R(\Gamma) \subseteq G$, and suppose

$$\operatorname{Ind}_{z=w}(\Gamma) = 0$$

for every $w \in \mathbb{C} \setminus G$. If $f : G \to \mathbb{C}$ is holomorphic, then

$$\int_\Gamma f(z)\,\mathrm{d}z = 0$$

Proof: Set

$$K = R(\Gamma) \cup \{z \in \mathbb{C} \setminus R(\Gamma) \mid \operatorname{Ind}_{z=z}(\Gamma) \neq 0\}.$$

By Local Constancy of the Winding Number (7.7) and Vanishing on the Unbounded Component (7.8), the winding number is locally constant on $\mathbb{C} \setminus R(\Gamma)$ and is 0 on its unbounded component. Thus the second set is a bounded union of connected components of $\mathbb{C} \setminus R(\Gamma)$. Hence K is compact. By the hypothesis on winding numbers outside G, we have $K \subseteq G$.

By Separation Lemma (7.10), there is a simple contour Γ' such that

$$R(\Gamma') \subseteq G \setminus K \quad \text{and} \quad K \subseteq \operatorname{int}(\Gamma').$$

Applying Cauchy's Formula for a Separating Contour (7.11) to this separating contour, for every $z \in K$ we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

In particular, this formula holds for every $z \in R(\Gamma)$. Therefore

$$\int_\Gamma f(z)\,\mathrm{d}z = \frac{1}{2\pi i} \int_\Gamma \left(\int_{\Gamma'} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \right) \mathrm{d}z.$$

Since both contours are finite sums of piecewise-C^1 curves and the integrand is continuous on $R(\Gamma') \times R(\Gamma)$, we may interchange the two integrals:

$$\int_\Gamma f(z)\,\mathrm{d}z = \frac{1}{2\pi i} \int_{\Gamma'} f(\zeta) \left(\int_\Gamma \frac{1}{\zeta - z}\,\mathrm{d}z \right) \mathrm{d}\zeta.$$

For $\zeta \in R(\Gamma')$, we have $\zeta \notin K$, so $\zeta \notin R(\Gamma)$ and $\operatorname{Ind}_{z=\zeta}(\Gamma) = 0$. Hence

$$\int_\Gamma \frac{1}{\zeta - z}\,\mathrm{d}z = -\int_\Gamma \frac{1}{z - \zeta}\,\mathrm{d}z = -2\pi i \operatorname{Ind}_{z=\zeta}(\Gamma) = 0.$$

Substituting this into the preceding expression gives

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

□

Note: We now see that our old results easily follow from this generalized result:

Corollary 7.13 **Cauchy's Theorem for Convex Regions**

Let $G \subseteq \mathbb{C}$ be convex, let $f : G \to \mathbb{C}$ be holomorphic, and let Γ be a piecewise-C^1 closed curve in G. Then $\operatorname{int}(\Gamma) \subseteq G$, and hence

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

Proof: This is Cauchy's Theorem for Convex Regions (4.8). □

Corollary 7.14 **Cauchy's Theorem for the Annulus**

It follows from Generalized Cauchy Theorem (7.12).

7.6 Residues at Isolated Singularities

Definition 7.8

Suppose f is holomorphic with an isolated singularity $z_0 \in \mathbb{C}$. Denote the Laurent series of f as $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. We define the residue of f to be a_{-1} and denote it as $\operatorname{res}_{z=z_0}(f(z))$, $\operatorname{res}(f, z_0)$, or $\operatorname{res}_{z_0}(f)$.

Proposition 7.15 **Simple-Pole Quotient Criterion**

Suppose g and h are holomorphic near z_0, with

$$h(z_0) = 0, \quad h'(z_0) \neq 0, \quad \text{and} \quad g(z_0) \neq 0.$$

Then $\frac{g}{h}$ has a simple pole at z_0.

Proof: Since $h(z_0) = 0$ and $h'(z_0) \neq 0$, the function h has a simple zero at z_0. Thus we may write

$$h(z) = (z - z_0)k(z),$$

where k is holomorphic near z_0 and $k(z_0) = h'(z_0) \neq 0$. Therefore

$$\frac{g(z)}{h(z)} = \frac{1}{z - z_0} \cdot \frac{g(z)}{k(z)}.$$

Since $k(z_0) \neq 0$, the function $\frac{g}{k}$ is holomorphic near z_0. Also $\left(\frac{g}{k}\right)(z_0) \neq 0$ because $g(z_0) \neq 0$. Hence $\frac{g}{h}$ has a simple pole at z_0. □

Theorem 7.16 **Residue Formula at a Pole**

Suppose z_0 is a pole of order m of f. Then

$$\operatorname{res}_{z=z_0}(f) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}((z-z_0)^m f(z)).$$

Proof: Since z_0 is a pole of order m, we may write

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$$

near z_0, with $a_{-m} \neq 0$. Hence

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n.$$

The coefficient of $(z-z_0)^{m-1}$ in this Taylor series is a_{-1}. Therefore

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}((z-z_0)^m f(z)).$$

Since $\operatorname{res}_{z=z_0}(f) = a_{-1}$, the formula follows. □

Corollary 7.17 **Simple-Pole Residue Formula**

Suppose z_0 is a simple pole of f. Then

$$\operatorname{res}_{z=z_0}(f) = \lim_{z \to z_0} (z-z_0) f(z).$$

Proof: This is Residue Formula at a Pole (7.16) with $m = 1$. □

Exercise 7.6.1

Compute the residues of the following functions at each of their isolated singularities:

1. $$e^{\frac{1}{z}}$$
2. $$\frac{z^3}{z-1}$$
3. $$\frac{z^3}{(z-1)^3}$$
4. $$\frac{z^k}{(z-a)(z-b)^2}$$

   for $k \geq 0$, $a \neq b$.
5. $$\frac{\sin z}{2z^2 - 5z + 2}$$
6. $$\frac{2z^2 - 5z + 2}{\sin z}$$

7.7 The Residue Theorem

Note

$a_n = \frac{1}{2\pi i} \int_{C_r} (z - z_0)^{-n-1} f(z)\, \mathrm{d}z$ and $\int_{C_r} f(z)\, \mathrm{d}z = 2\pi i \operatorname{res}_{z_0} f$.

Theorem 7.18 **Residue Theorem**

Let $G \subseteq \mathbb{C}$ be open, let $z_1, \ldots, z_p \in G$, and let f be holomorphic on $G \setminus \{z_1, \ldots, z_p\}$, where $z_1, \ldots, z_p$ are isolated singularities of f. Let Γ be a contour such that

$$R(\Gamma) \subseteq G \setminus \{z_1, \ldots, z_p\} \quad \text{and} \quad \operatorname{Ind}_{z=w}(\Gamma) = 0 \quad \text{for every} \quad w \in \mathbb{C} \setminus G.$$

Then

$$\int_\Gamma f(z)\, \mathrm{d}z = 2\pi i \sum_{j=1}^{p} \operatorname{Ind}_{z=z_j}(\Gamma) \operatorname{res}_{z=z_j}(f).$$

Proof: Choose $r > 0$ so small that the closed discs $\overline{D(z_j, r)}$ are pairwise disjoint, lie in G, and do not meet $R(\Gamma)$. Let C_j be the positively oriented circle $C_r(z_j) = \partial D(z_j, r)$, and set

$$n_j = \operatorname{Ind}_{z=z_j}(\Gamma) \quad \text{and} \quad \Lambda = \Gamma - \sum_{j=1}^{p} n_j C_j.$$

We apply Generalized Cauchy Theorem (7.12) to Λ in $U = G \setminus \{z_1, \ldots, z_p\}$. If $w \notin G$, then each C_j has winding number 0 about w, so $\operatorname{Ind}_{z=w}(\Lambda) = \operatorname{Ind}_{z=w}(\Gamma) = 0$. Also, for each k,

$$\operatorname{Ind}_{z=z_k}(\Lambda) = \operatorname{Ind}_{z=z_k}(\Gamma) - n_k = 0,$$

since $\operatorname{Ind}_{z=z_k}(C_j) = 1$ when $j = k$ and 0 otherwise. Thus $\operatorname{Ind}_{z=w}(\Lambda) = 0$ for every $w \in \mathbb{C} \setminus U$. Since f is holomorphic on U, Generalized Cauchy Theorem (7.12) gives

$$0 = \int_\Lambda f(z)\,\mathrm{d}z = \int_\Gamma f(z)\,\mathrm{d}z - \sum_{j=1}^{p} n_j \int_{C_j} f(z)\,\mathrm{d}z.$$

For each j, write the Laurent expansion of f near z_j as

$$f(z) = \sum_{n=-\infty}^{\infty} a_{j,n}(z - z_j)^n.$$

This series converges uniformly on C_j, so Interchange of Limit and Path Integral (4.6) allows us to integrate term by term. The integral of $(z - z_j)^n$ around C_j is 0 unless $n = -1$, and it is $2\pi i$ when $n = -1$. Hence

$$\int_{C_j} f(z)\,\mathrm{d}z = 2\pi i a_{j,-1} = 2\pi i \operatorname{res}_{z=z_j}(f).$$

Substituting this into the previous equality gives

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{p} \operatorname{Ind}_{z=z_j}(\Gamma) \operatorname{res}_{z=z_j}(f).$$

$\square$

Corollary 7.19 **Cauchy's Integral Formula**

Let $G \subseteq \mathbb{C}$ be open, let Γ be a simple contour with $R(\Gamma) \cup \operatorname{int}(\Gamma) \subseteq G$, and let $f : G \to \mathbb{C}$ be holomorphic. Then

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for every $z \in \operatorname{int}(\Gamma)$.

7.8 Computing Integrals by Residues

Exercise 7.8.1

Compute

$$\int_{C_1} (z-2)^{-1}(2z+1)^{-2}(3z-1)^{-3}\,\mathrm{d}z$$

Exercise 7.8.2

Compute $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$. Hint: Consider $f(z) = \frac{e^{iz}}{z}$.

Exercise 7.8.3 **An Improper Integral via a Sector Contour**

Compute

$$\int_0^\infty \frac{1}{1+x^a}\,\mathrm{d}x$$

for $a > 1$.

Proof: Set $\alpha = 2\frac{\pi}{a}$. Choose $\delta > 0$ such that $\alpha + 2\delta < 2\pi$, and use the branch of logarithm with argument in $(-\delta, \alpha + \delta)$. On this branch, define

$$z^a = e^{a\,\mathrm{Log}(z)} \quad \text{and} \quad f(z) = \frac{1}{1+z^a}.$$

The sector $0 \leq \arg(z) \leq \alpha$ contains exactly one pole of f, namely

$$z_0 = e^{i\frac{\pi}{a}}.$$

Since $(z^a)' = az^{a-1}$, this pole is simple and

$$\mathrm{res}_{z=z_0}(f) = \frac{1}{az_0^{a-1}} = -\frac{z_0}{a}.$$

Let $\Gamma_{\varepsilon,R}$ be the positively oriented boundary of the truncated sector

$$\left\{re^{i\theta} \mid \varepsilon \leq r \leq R, 0 \leq \theta \leq \alpha\right\}.$$

Along the positive real segment, the contour integral is

$$I_{\varepsilon,R} := \int_\varepsilon^R \frac{1}{1+x^a}\,\mathrm{d}x.$$

Along the ray of argument α, write $z = e^{i\alpha}x$ and use $\left(e^{i\alpha}x\right)^a = x^a$. Because this ray is traversed from R to ε, its contribution is $-e^{i\alpha}I_{\varepsilon,R}$. The inner circular arc has integral tending to 0 as $\varepsilon \to 0$, while the outer circular arc has integral $O(R^{1-a})$, which tends to 0 because $a > 1$.

The Residue Theorem therefore gives, after letting $\varepsilon \to 0$ and $R \to \infty$,

$$\left(1 - e^{i\alpha}\right)\int_0^\infty \frac{1}{1+x^a}\,\mathrm{d}x = -2\pi\frac{i}{a}e^{i\frac{\pi}{a}}.$$

Since $\alpha = 2\frac{\pi}{a}$ and

$$1 - e^{2\pi \frac{i}{a}} = -2ie^{i\frac{\pi}{a}} \sin\left(\frac{\pi}{a}\right),$$

we obtain

$$\int_0^\infty \frac{1}{1+x^a} \, \mathrm{d}x = \frac{\pi}{a} \csc\left(\frac{\pi}{a}\right).$$

□

7.9 Simply Connected Regions and Global Primitives

Definition 7.9 **Homotopy of Paths**

Let $G \subseteq \mathbb{C}$, and let $\gamma_0, \gamma_1 : [0,1] \to G$ be paths with the same initial point z_0 and terminal point z_1. A *homotopy from* γ_0 *to* γ_1 *in* G *relative to their endpoints* is a continuous map

$$H : [0,1]^2 \to G$$

such that, for every $s, t \in [0,1]$,

$$H(s,0) = \gamma_0(s), \quad H(s,1) = \gamma_1(s),$$

and

$$H(0,t) = z_0, \quad H(1,t) = z_1.$$

If such an H exists, then γ_0 and γ_1 are *homotopic in* G *relative to their endpoints.*

Definition 7.10 **Null-Homotopic Closed Curve**

Let $\gamma : [0,1] \to G$ be a closed curve based at $z_0 = \gamma(0) = \gamma(1)$. Then γ is *null-homotopic in* G if it is homotopic relative to its endpoints to the constant curve

$$c_{z_0}(s) = z_0.$$

Equivalently, γ represents the identity element $\left[c_{z_0}\right]$ of the fundamental group $\pi_1(G, z_0)$.

Definition 7.11 — **One-Point Compactification**

We write

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

and equip it with the *one-point compactification topology*. The usual open subsets of $\mathbb{C}$ remain open, while a neighbourhood of ∞ is any set containing a set of the form

$$\{\infty\} \cup (\mathbb{C} \setminus K),$$

where $K \subseteq \mathbb{C}$ is compact. Thus a path $\gamma : [0,1] \to \hat{\mathbb{C}}$ with $\gamma(1) = \infty$ is continuous at 1 whenever

$$|\gamma(t)| \to \infty \quad \text{as} \quad t \to 1^-.$$

Definition 7.12 — **Simply Connected**

Let $G \subseteq \mathbb{C}$ be a region. The *extended complement* of G is

$$\hat{\mathbb{C}} \setminus G.$$

We say that G is *simply connected* if its extended complement is connected.

Lemma 7.20 — **Polygonal Path-Connectedness of Regions**

Let $G \subseteq \mathbb{C}$ be open and connected. If $z_0, z \in G$, then there is a piecewise-$\mathcal{C}^1$ curve $\gamma : [0,1] \to G$ such that

$$\gamma(0) = z_0 \quad \text{and} \quad \gamma(1) = z.$$

Proof: Fix $z_0 \in G$, and let A be the set of points in G that can be joined to z_0 by a polygonal path in G. Clearly $z_0 \in A$. If $z \in A$, choose $r > 0$ such that $D(z,r) \subseteq G$. Every point of $D(z,r)$ can be joined to z by a line segment in $D(z,r)$, and hence to z_0 by a polygonal path in G. Thus $D(z,r) \subseteq A$, so A is open in G.

If $z \in G \setminus A$, choose $r > 0$ such that $D(z,r) \subseteq G$. If some $w \in D(z,r)$ belonged to A, adjoining the line segment from w to z would give a polygonal path from z_0 to z, a contradiction. Therefore $D(z,r) \subseteq G \setminus A$, so $G \setminus A$ is also open in G. Since G is connected and A is nonempty, open, and closed in G, we have $A = G$. A polygonal path is piecewise-$\mathcal{C}^1$, which proves the result. □

Example

Set

$$S_+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1\} \quad \text{and} \quad S_- = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < -1\},$$

and let

$$S = S_+ \cup S_- \cup \{\infty\}.$$

We show that S is path-connected. For $z \in S_+$, define

$$\gamma_z^+(t) = \begin{cases} z + \frac{t}{1-t} & 0 \le t < 1 \\ \infty & t = 1. \end{cases}$$

For $t < 1$, we have $\mathrm{Re}(\gamma_z^+(t)) > 1$, so the path remains in S_+, and

$$|\gamma_z^+(t)| \to \infty \quad \text{as} \quad t \to 1^-.$$

By the definition of the one-point compactification topology, γ_z^+ is continuous at 1. Hence it is a path in $S_+ \cup \{\infty\}$ from z to ∞.

Similarly, for $z \in S_-$, the map

$$\gamma_z^-(t) = \begin{cases} z - \frac{t}{1-t} & 0 \le t < 1 \\ \infty & t = 1 \end{cases}$$

is a path in $S_- \cup \{\infty\}$ from z to ∞. Thus every point of S can be joined to ∞ by a path in S, so S is path-connected and hence connected. Notice that $S \setminus \{\infty\} = S_+ \cup S_-$ is disconnected in the usual topology on $\mathbb{C}$.

Theorem 7.21 **Characterizations of Simply Connected Regions**

Let $G \subseteq \mathbb{C}$ be a region. The following are equivalent:

1. G is simply connected.
2. If Γ is a contour in G, then
$$\mathrm{Ind}_{z=w}(\Gamma) = 0$$
for every $w \in \mathbb{C} \setminus G$.
3. If Γ is a contour in G, and $f : G \to \mathbb{C}$ is holomorphic, then $\int_\Gamma f(z)\,\mathrm{d}z = 0$.
4. If $f : G \to \mathbb{C}$ is holomorphic, then f has a primitive.
5. If $f : G \to \mathbb{C} \setminus \{0\}$ is holomorphic, then there is a branch of log of f.
6. Every closed curve in G is null-homotopic in G.

Note

The implications

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$$

are proved directly below. Riemann Mapping Theorem (7.22) gives $(1) \Rightarrow (6)$, while $(6) \Rightarrow (5)$ follows from the lifting property of the covering map $\exp : \mathbb{C} \to \mathbb{C}^*$.

Proof of Characterizations of Simply Connected Regions: We first prove the cycle of implications among $(1)-(5)$.

$(1) \Rightarrow (2)$: Let Γ be a contour in G, and set

$$A = \hat{\mathbb{C}} \setminus G.$$

Since $R(\Gamma) \subseteq G$, we have

$$A \subseteq \hat{\mathbb{C}} \setminus R(\Gamma).$$

By (1), A is connected, and $\infty \in A$. Therefore A is contained in the connected component of $\hat{\mathbb{C}} \setminus R(\Gamma)$ containing ∞. Equivalently, every point of $\mathbb{C} \setminus G$ lies in the unbounded component of $\mathbb{C} \setminus R(\Gamma)$. By Vanishing on the Unbounded Component (7.8), the winding number of Γ is 0 on this component, so

$$\operatorname{Ind}_{z=w}(\Gamma) = 0$$

for every $w \in \mathbb{C} \setminus G$, so condition (2) holds.

$(2) \Rightarrow (3)$: Let Γ be a contour in G, and let $f : G \to \mathbb{C}$ be holomorphic. By (2), $\operatorname{Ind}_{z=w}(\Gamma) = 0$ for every $w \in \mathbb{C} \setminus G$. Since $R(\Gamma) \subseteq G$, Generalized Cauchy Theorem (7.12) gives

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

$(3) \Rightarrow (4)$: Let $f : G \to \mathbb{C}$ be holomorphic, and fix $z_0 \in G$. By Polygonal Path-Connectedness of Regions (7.20), for each $z \in G$ there is a piecewise-C^1 curve γ_z in G from z_0 to z. Define

$$F(z) = \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta.$$

This is well-defined. Indeed, if γ_1 and γ_2 are two such curves, traverse γ_1 and then traverse γ_2 in reverse. The resulting curve Γ is a closed piecewise-C^1 curve in G, so by (3),

$$0 = \int_\Gamma f(\zeta)\,\mathrm{d}\zeta = \int_{\gamma_1} f(\zeta)\,\mathrm{d}\zeta - \int_{\gamma_2} f(\zeta)\,\mathrm{d}\zeta.$$

Hence both choices give the same value of $F(z)$.

We now show that $F' = f$. Fix $z \in G$, and choose $r > 0$ such that $D(z, r) \subseteq G$. If $0 < |h| < r$, use a curve from z_0 to z followed by the line segment $\sigma_h(t) = z + th$, $0 \leq t \leq 1$, to compute $F(z+h)$. By well-definedness,

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{\sigma_h} f(\zeta)\,\mathrm{d}\zeta = \int_0^1 f(z+th)\,\mathrm{d}t.$$

Therefore,

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| \leq \int_0^1 |f(z+th)-f(z)|\,\mathrm{d}t.$$

The right-hand side tends to 0 as $h \to 0$ because f is continuous at z. Thus $F'(z) = f(z)$. Since $z \in G$ was arbitrary, F is a primitive of f on G.

(4) $\Rightarrow$ (5): Let $f : G \to \mathbb{C} \setminus \{0\}$ be holomorphic. Since f never vanishes, the function

$$h = \frac{f'}{f}$$

is holomorphic on G. By (4), there is a holomorphic function $H : G \to \mathbb{C}$ such that

$$H' = h = \frac{f'}{f}.$$

Consider the holomorphic function fe^{-H}. By the product and chain rules,

$$\left(fe^{-H}\right)' = f'e^{-H} - fH'e^{-H} = e^{-H}\left(f' - f\frac{f'}{f}\right) = 0.$$

Since G is connected, fe^{-H} is constant on G. Thus there is some $c \in \mathbb{C} \setminus \{0\}$ such that

$$f(z)e^{-H(z)} = c$$

for every $z \in G$. Choose $a \in \mathbb{C}$ such that $e^a = c$, and define

$$L = H + a.$$

Then L is holomorphic on G and

$$e^{L(z)} = e^{H(z)}e^a = ce^{H(z)} = f(z)$$

for every $z \in G$. Therefore L is a branch of $\log f$ on G.

(5) $\Rightarrow$ (1): Suppose, toward a contradiction, that

$$K = \hat{\mathbb{C}} \setminus G$$

is disconnected. Since K is compact, there are disjoint nonempty compact sets A and B such that

$$K = A \cup B.$$

Relabel them if necessary so that $\infty \in B$. Set

$$U = \hat{\mathbb{C}} \setminus B.$$

Since $\infty \in B$, the set U is an open subset of $\mathbb{C}$, and A is a compact subset of U. By Separation Lemma (7.10), there is a simple contour Γ such that

$$R(\Gamma) \subseteq U \setminus A = G \quad \text{and} \quad A \subseteq \operatorname{int}(\Gamma).$$

Choose $w_0 \in A$, and define

$$f : G \to \mathbb{C} \setminus \{0\}, \quad f(z) = z - w_0.$$

This function is holomorphic and never vanishes on G because $w_0 \notin G$. By (5), it has a holomorphic branch of logarithm $L : G \to \mathbb{C}$, so

$$e^{L(z)} = z - w_0.$$

Differentiating gives

$$L'(z)e^{L(z)} = 1,$$

and hence

$$L'(z) = \frac{1}{z - w_0}.$$

Since L is a primitive of $\frac{1}{z-w_0}$ on G and Γ is a contour in G, Fundamental Theorem of Calculus (For Path Integrals) (4.4) gives

$$\int_\Gamma \frac{1}{z - w_0} \, \mathrm{d}z = 0.$$

On the other hand, $w_0 \in A \subseteq \operatorname{int}(\Gamma)$, so $\operatorname{Ind}_{z=w_0}(\Gamma) = 1$. Therefore,

$$\int_\Gamma \frac{1}{z - w_0} \, \mathrm{d}z = 2\pi i \operatorname{Ind}_{z=w_0}(\Gamma) = 2\pi i,$$

a contradiction. Thus $\hat{\mathbb{C}} \setminus G$ is connected.

(1) $\Rightarrow$ (6): Let $\gamma : [0, 1] \to G$ be a closed curve. By Riemann Mapping Theorem (7.22), either $G = \mathbb{C}$ or there is a biholomorphism $\varphi : G \to \mathbb{D}$.

If $G = \mathbb{C}$, define

$$H(s, t) = (1 - t)\gamma(s) + t\gamma(0).$$

Since $\mathbb{C}$ is convex, H is a null-homotopy of γ in $\mathbb{C}$.

Otherwise, define

$$K(s, t) = (1 - t)(\varphi \circ \gamma)(s) + t(\varphi \circ \gamma)(0).$$

Since $\mathbb{D}$ is convex, K is a null-homotopy of $\varphi \circ \gamma$ in $\mathbb{D}$. Therefore

$$H = \varphi^{-1} \circ K$$

is a null-homotopy of γ in G. Hence every closed curve in G is null-homotopic in G.

(6) $\Rightarrow$ (5): Let $f : G \to \mathbb{C}^*$ be holomorphic. Fix $z_0 \in G$ and choose $a \in \mathbb{C}$ such that $e^a = f(z_0)$. By (6), every closed curve in G is null-homotopic, so its image under f is null-homotopic in $\mathbb{C}^*$. The lifting property of the covering map

$$\exp : \mathbb{C} \to \mathbb{C}^*$$

therefore gives a continuous lift $L : G \to \mathbb{C}$ such that

$$L(z_0) = a \quad \text{and} \quad e^{L(z)} = f(z)$$

for every $z \in G$.

The lift L is holomorphic. Indeed, near each point of G, the image of f lies in a small disc in $\mathbb{C}^*$ on which there is a holomorphic branch ℓ of logarithm. On a sufficiently small connected neighbourhood,

$$L - (\ell \circ f)$$

is continuous and takes values in $2\pi i\mathbb{Z}$, so it is constant. Thus L differs locally from the holomorphic function $\ell \circ f$ by a constant. Therefore L is holomorphic on G and is a branch of $\log f$. □

Theorem 7.22 **Riemann Mapping Theorem**

Let $G \subseteq \mathbb{C}$ be a nonempty region such that $\hat{\mathbb{C}} \setminus G$ is connected. Then either $G = \mathbb{C}$ or there exists a biholomorphism

$$\varphi : G \to \mathbb{D}.$$

That is, φ is a holomorphic bijection whose inverse is also holomorphic.

Proof: Out of scope. □

8 Linear-Fractional Transformations of the Riemann Sphere

8.1 The Riemann Sphere as a Projective Line

Definition 8.1 **Riemann Sphere**

The **Riemann sphere** is

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

It is equipped with the one-point compactification topology: the usual open subsets of $\mathbb{C}$ are open in $\hat{\mathbb{C}}$, and a neighbourhood of ∞ is any set containing a set of the form

$$\{\infty\} \cup (\mathbb{C} \setminus K),$$

where $K \subseteq \mathbb{C}$ is compact.

Definition 8.2 **Projective Line**

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. On $\mathbb{F}^2 \setminus \{(0,0)\}$, define

$$(x_0, x_1) \sim (y_0, y_1) \Leftrightarrow (y_0, y_1) = \lambda(x_0, x_1)$$

for some $\lambda \in \mathbb{F} \setminus \{0\}$. The **projective line over** $\mathbb{F}$ is the quotient

$$\mathbb{FP}^1 = (\mathbb{F}^2 \setminus \{(0,0)\}) / \sim .$$

The equivalence class of (x_0, x_1) is denoted by $[x_0 : x_1]$. These are called **homogeneous coordinates** because

$$[x_0 : x_1] = [\lambda x_0 : \lambda x_1]$$

for every $\lambda \in \mathbb{F} \setminus \{0\}$. Equivalently, $\mathbb{FP}^1$ is the set of one-dimensional $\mathbb{F}$-linear subspaces of $\mathbb{F}^2$.

Note

In particular, $\mathbb{RP}^1$ is the space of lines through the origin in $\mathbb{R}^2$. Moreover,

$$\mathbb{RP}^1 \cong S^1 \cong \hat{\mathbb{R}},$$

where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}$.

For $\mathbb{RP}^1 \cong \hat{\mathbb{R}}$, take a line $[x : y]$ through the origin in $\mathbb{R}^2$ and map it to where it intersects the line 1 unit above the x-axis, namely $\frac{x}{y}$ or to ∞, if $y = 0$.

For $S^1 \cong \hat{\mathbb{R}}$, perform a stereographic projection from the north pole $(0,1)$ onto the x-axis, and $(0,1)$ itself to ∞.

Proposition 8.1 **Riemann Sphere as the Complex Projective Line**

The map

$$\iota : \hat{\mathbb{C}} \to \mathbb{CP}^1, \quad \iota(z) = [z : 1], \quad \iota(\infty) = [1 : 0],$$

is a bijection. When $\mathbb{CP}^1$ has the quotient topology, ι is a homeomorphism.

Proof: Every point of $\mathbb{CP}^1$ has the form $[z_0 : z_1]$. If $z_1 \neq 0$, then

$$[z_0 : z_1] = \left[\frac{z_0}{z_1} : 1\right] = \iota\left(\frac{z_0}{z_1}\right).$$

If $z_1 = 0$, then $z_0 \neq 0$ and $[z_0 : 0] = [1 : 0] = \iota(\infty)$. These two forms are unique, so ι is bijective.

On the chart

$$U_1 = \{[z_0 : z_1] : z_1 \neq 0\},$$

the coordinate map $[z_0 : z_1] \to \frac{z_0}{z_1}$ identifies U_1 with $\mathbb{C}$. On the chart $U_0 = \{[z_0 : z_1] : z_0 \neq 0\}$, the coordinate map $[z_0 : z_1] \to \frac{z_1}{z_0}$ identifies a neighbourhood of $[1 : 0]$ with a neighbourhood of 0. Under ι^{-1}, the set where $\left|\frac{z_1}{z_0}\right| < \varepsilon$ corresponds to

$$\{\infty\} \cup \left\{z \in \mathbb{C} : |z| > \frac{1}{\varepsilon}\right\},$$

which is a neighbourhood of ∞ in the one-point compactification topology. Thus ι and ι^{-1} are continuous. $\square$

We use this homeomorphism to identify $\mathbb{CP}^1$ with $\hat{\mathbb{C}}$.

8.2 Matrix Representatives and the Projective Linear Group

Proposition 8.2 **Projectivization of Linear Automorphisms**

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C}).$$

Then A induces a well-defined bijection

$$\Phi_A : \mathbb{CP}^1 \to \mathbb{CP}^1, \quad \Phi_A([v]) = [Av].$$

Under the identification $\mathbb{CP}^1 \cong \hat{\mathbb{C}}$, this map is given by

$$\Phi_A(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } \ z \in \mathbb{C} \ \text{ and } \ cz+d \neq 0 \\ \infty & \text{if } \ z \in \mathbb{C} \ \text{ and } \ cz+d = 0 \\ \frac{a}{c} & \text{if } \ z = \infty \ \text{ and } \ c \neq 0 \\ \infty & \text{if } \ z = \infty \ \text{ and } \ c = 0. \end{cases}$$

Proof: If $[v] = [\lambda v]$ for $\lambda \neq 0$, then

$$[A(\lambda v)] = [\lambda A v] = [Av],$$

so the map is independent of the representative. Since A is invertible, $Av \neq 0$ whenever $v \neq 0$, and $\Phi_{A^{-1}}$ is the inverse of Φ_A.

For $z \in \mathbb{C}$,

$$\Phi_A([z : 1]) = [az + b : cz + d].$$

If $cz + d \neq 0$, this is $\left[\frac{az+b}{cz+d} : 1\right]$. If $cz + d = 0$, then $az + b \neq 0$, since otherwise $A(z, 1)^T = (0, 0)^T$, contradicting the invertibility of A; hence the image is $[1 : 0]$. Finally,

$$\Phi_A([1 : 0]) = [a : c],$$

which gives the two stated cases at ∞. □

Definition 8.3 **Linear-Fractional Transformation**

A **linear-fractional transformation**, also called a **Möbius transformation**, is a map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $\varphi = \Phi_A$ for some $A \in GL_2(\mathbb{C})$. Thus, away from its possible pole,

$$\varphi(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

with the values at the pole and at ∞ defined as in Projectivization of Linear Automorphisms (8.2).

Note

If $c \neq 0$, then $-\frac{d}{c}$ is the unique pole of φ and

$$\varphi\left(-\frac{d}{c}\right) = \infty, \quad \varphi(\infty) = \frac{a}{c}.$$

If $c = 0$, then φ is the affine map

$$\varphi(z) = \frac{a}{d}z + \frac{b}{d}$$

on $\mathbb{C}$ and fixes ∞.

Theorem 8.3 **Projective Linear Group of the Riemann Sphere**

The linear-fractional transformations form a group under composition. The map

$$\rho : GL_2(\mathbb{C}) \to \text{Möb}\left(\hat{\mathbb{C}}\right), \quad \rho(A) = \Phi_A,$$

is a surjective group homomorphism with kernel

$$\ker(\rho) = \{\lambda I_2 : \lambda \in \mathbb{C} \setminus \{0\}\}.$$

Consequently,

$$\text{Möb}\left(\hat{\mathbb{C}}\right) \cong GL_2(\mathbb{C}) / \{\lambda I_2 : \lambda \in \mathbb{C} \setminus \{0\}\},$$

and this quotient is denoted by $PGL_2(\mathbb{C})$.

Proof: For $A, B \in GL_2(\mathbb{C})$ and $[v] \in \mathbb{CP}^1$,

$$(\Phi_A \circ \Phi_B)([v]) = \Phi_A([Bv]) = [ABv] = \Phi_{AB}([v]).$$

Therefore $\rho(AB) = \rho(A) \circ \rho(B)$. Surjectivity follows from the definition of a linear-fractional transformation.

If $A = \lambda I_2$ with $\lambda \neq 0$, then $[Av] = [v]$ for every nonzero v, so $A \in \ker(\rho)$. Conversely, suppose Φ_A is the identity. Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since Φ_A fixes $[1 : 0]$ and $[0 : 1]$, we must have $c = 0$ and $b = 0$. Since it also fixes $[1 : 1]$, we have $[a : d] = [1 : 1]$, and hence $a = d$. Thus $A = aI_2$, with $a \neq 0$ because A is invertible. The claimed isomorphism now follows from the First Isomorphism Theorem. $\square$

Corollary 8.4 **Scalar Equivalence of Projective Representatives**

If $A, B \in GL_2(\mathbb{C})$, then $\Phi_A = \Phi_B$ if and only if

$$A = \lambda B$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Proof: We have $\Phi_A = \Phi_B$ exactly when $\Phi_{B^{-1}A}$ is the identity. By the kernel computation in Projective Linear Group of the Riemann Sphere (8.3), this holds exactly when $B^{-1}A = \lambda I_2$ for some $\lambda \neq 0$, or equivalently when $A = \lambda B$. □

Note

Over $\mathbb{C}$, this group may also be written as

$$PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I_2\}.$$

Indeed, if $A \in GL_2(\mathbb{C})$, choose $\mu \in \mathbb{C} \setminus \{0\}$ such that $\mu^2 = \det(A)^{-1}$. Then $\mu A \in SL_2(\mathbb{C})$ and induces the same linear-fractional transformation as A. The two choices of μ differ by a sign, which accounts for the quotient by $\{\pm I_2\}$.

Corollary 8.5 **Inverse Linear-Fractional Transformation**

If $\varphi = \Phi_A$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\varphi^{-1}(z) = \frac{dz - b}{-cz + a},$$

with the usual conventions at its pole and at ∞.

Proof: Since scalar multiples induce the same projective map, the factor $\frac{1}{ad-bc}$ in

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

may be omitted. The result follows from $\Phi_A^{-1} = \Phi_{A^{-1}}$. □

8.3 Biholomorphisms, Three-Point Rigidity, and Fixed Points

Proposition 8.6 **Linear-Fractional Transformations are Biholomorphisms**

Every linear-fractional transformation is a biholomorphism of the Riemann sphere. At each finite point other than its pole,

$$\varphi'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0.$$

Proof: The derivative formula follows from the quotient rule. At the pole $z_0 = -\frac{d}{c}$, use the target coordinate $w \to \frac{1}{w}$ about ∞; in this coordinate the map is

$$\frac{1}{\varphi(z)} = \frac{cz + d}{az + b},$$

which is holomorphic near z_0 because $az_0 + b \neq 0$.

To check holomorphicity at ∞, use the source coordinate $w = \frac{1}{z}$. If $c \neq 0$, then

$$\varphi\left(\frac{1}{w}\right) = \frac{a+bw}{c+dw}$$

is holomorphic near $w = 0$. If $c = 0$, then $\varphi(\infty) = \infty$, and in the target coordinate at ∞ we have

$$\frac{1}{\varphi\left(\frac{1}{w}\right)} = \frac{dw}{a+bw},$$

which is holomorphic near $w = 0$. Hence φ is holomorphic everywhere on $\hat{\mathbb{C}}$. Its inverse is also a linear-fractional transformation by Inverse Linear-Fractional Transformation (8.5), so φ is biholomorphic. □

Proposition 8.7 **Sharp Three-Transitivity**

Given two ordered triples of distinct points

$$z_1, z_2, z_3 \in \hat{\mathbb{C}} \quad \text{and} \quad w_1, w_2, w_3 \in \hat{\mathbb{C}},$$

there is a unique linear-fractional transformation φ such that

$$\varphi(z_j) = w_j$$

for $j = 1, 2, 3$.

Proof: Regard the points as lines in $\mathbb{C}^2$. Choose nonzero representatives v_1 and v_3 of z_1 and z_3. Since these points are distinct, v_1, v_3 is a basis. A representative of z_2 therefore has the form $\alpha v_1 + \beta v_3$, where $\alpha, \beta \neq 0$; after rescaling v_1 and v_3, we may take $v_1 + v_3$ as a representative of z_2. Similarly, choose representatives u_1, u_3 of w_1, w_3 so that $u_1 + u_3$ represents w_2. The unique linear map A satisfying

$$Av_1 = u_1, \quad Av_3 = u_3$$

is invertible and induces a linear-fractional transformation with the desired values.

For uniqueness, suppose Φ_A and Φ_B both have the desired values. Then $\Phi_{B^{-1}A}$ fixes the three distinct lines represented by v_1, v_3, and $v_1 + v_3$. Fixing the first two lines makes the matrix of $B^{-1}A$ diagonal in the basis v_1, v_3; fixing the third forces its two diagonal entries to agree. Thus $B^{-1}A$ is scalar, so $\Phi_A = \Phi_B$. □

Corollary 8.8 **Rigidity from Three Fixed Points**

A linear-fractional transformation that fixes three distinct points of $\hat{\mathbb{C}}$ is the identity.

Proposition 8.9 **Fixed-Point Classification**

A nonidentity linear-fractional transformation has exactly one or two fixed points in $\hat{\mathbb{C}}$.

Proof: Let $\varphi = \Phi_A$. Under the identification $\hat{\mathbb{C}} \cong \mathbb{CP}^1$, a point $[v]$ is fixed precisely when

$$[Av] = [v],$$

which holds precisely when v is an eigenvector of A. Thus the fixed points of φ are exactly the eigenlines of A.

The characteristic polynomial of A has degree two and splits over $\mathbb{C}$. If it has two distinct roots, then A has exactly two eigenlines. If it has one repeated root, then either its eigenspace is one-dimensional, giving one eigenline, or its eigenspace is two-dimensional, in which case A is scalar. The latter would make φ the identity, contrary to the hypothesis. Therefore φ has exactly one or two fixed points. □

9 Harmonic Functions

9.1 The Laplacian and Holomorphic Functions

Definition 9.1

$G \subseteq \mathbb{C} \cong \mathbb{R}^2$ open, $f : G \to \mathbb{C}$ is *harmonic* if it is in $\mathcal{C}^2$ (which means that the second order partials are continuous) and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \text{ (Laplacian Equation)}$$

Note

f is harmonic if and only if $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$ are harmonic.

Example

$f(x, y) = x^2 - y^2$, $g(x, y) = 2xy$ are harmonic functions.

Proposition 9.1 **Holomorphic Functions are Harmonic**

If $f : G \to \mathbb{C}$ is holomorphic, then it is harmonic.

Proof: Write $f = u + iv$, where $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$. By Holomorphic Functions are Smooth (4.14), $u, v \in \mathcal{C}^2$. Cauchy-Riemann Equations (2.2) gives

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Differentiating the first equation with respect to x and the second with respect to y gives

$$u_{xx} = v_{yx} \quad \text{and} \quad u_{yy} = -v_{xy}.$$

By Clairaut's theorem, the mixed partials of v are equal, so

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Thus u is harmonic.

Similarly, differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x gives

$$v_{yy} = u_{xy} \quad \text{and} \quad v_{xx} = -u_{yx}.$$

By Clairaut's theorem, the mixed partials of u are equal, so

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$$

Therefore v is harmonic, and hence f is harmonic. □

9.2 Harmonic Conjugates and Simple Connectivity

Definition 9.2

Let $u : G \to \mathbb{R}$ be harmonic. A function $v : G \to \mathbb{R}$ is a *harmonic conjugate* of u if $u + iv$ is holomorphic on G.

Theorem 9.2 **Existence of Harmonic Conjugates**

If G is simply connected and $u : G \to \mathbb{R}$ is harmonic, then u has a harmonic conjugate.

Proof: Define

$$f = u_x - iu_y.$$

Write $f = p + iq$, where $p = u_x$ and $q = -u_y$. Since u is harmonic,

$$p_x = u_{xx} = -u_{yy} = q_y.$$

Moreover, Clairaut's theorem gives

$$q_x = -u_{yx} = -u_{xy} = -p_y.$$

Thus, by Cauchy-Riemann Equations (2.2), f is holomorphic.

By Characterizations of Simply Connected Regions (7.21), f has a primitive on G. Hence there is a holomorphic function $g : G \to \mathbb{C}$ such that $g' = f$. Fix $z_0 \in G$. After adding a constant to g, we may assume that $g(z_0) = u(z_0)$. Write

$$g = u_1 + iv.$$

Since g is holomorphic, Cauchy-Riemann Equations (2.2) implies

$$g' = (u_1)_x - i(u_1)_y.$$

Comparing this with

$$g' = f = u_x - iu_y$$

gives

$$(u_1)_x = u_x \quad \text{and} \quad (u_1)_y = u_y.$$

Therefore $u_1 - u$ has zero gradient on the connected set G, so it is constant. Since

$$(u_1 - u)(z_0) = \operatorname{Re}(g(z_0)) - u(z_0) = 0,$$

we have $u_1 = u$. Consequently, $g = u + iv$ is holomorphic, and hence v is a harmonic conjugate of u. □

Example

Let

$$G = \{z \in \mathbb{C} \mid 0 < |z| < R\} \quad \text{and} \quad u(x, y) = \frac{1}{2}\log(x^2 + y^2) = \log|z|.$$

A direct calculation gives

$$u_x = \frac{x}{x^2 + y^2} \quad \text{and} \quad u_y = \frac{y}{x^2 + y^2},$$

and hence $u_{xx} + u_{yy} = 0$. Thus u is harmonic on G.

Suppose that u had a harmonic conjugate v on G. Then $F = u + iv$ would be holomorphic, and Cauchy-Riemann Equations (2.2) would give

$$F' = u_x - iu_y = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

Thus F would be a primitive of $\frac{1}{z}$ on G. But if $C_r(t) = re^{it}$ for $0 \le t \le 2\pi$ and $0 < r < R$, then

$$\int_{C_r} \frac{1}{z}\,\mathrm{d}z = 2\pi i \ne 0,$$

contradicting Fundamental Theorem of Calculus (For Path Integrals) (4.4). Therefore u has no harmonic conjugate on the punctured disc G.

Corollary 9.3 Harmonic-Conjugate Characterization of Simple Connectivity

Let $G \subseteq \mathbb{C}$ be open and connected. Then G is simply connected if and only if every real-valued harmonic function on G admits a harmonic conjugate.

Proof: ($\Longrightarrow$) This is Existence of Harmonic Conjugates (9.2).

($\Longleftarrow$) Suppose every real-valued harmonic function on G admits a harmonic conjugate. Let $f : G \to \mathbb{C} \setminus \{0\}$ be holomorphic. The function

$$u = \log|f|$$

is harmonic: locally, f has a holomorphic logarithm, and u is the real part of that logarithm. By assumption, there is a harmonic function v such that $h = u + iv$ is holomorphic. Then

$$q = \frac{e^h}{f}$$

is holomorphic on G, and

$$|q| = \frac{e^u}{|f|} = 1.$$

Since G is connected and $|q| = 1$, the Open Mapping Theorem implies that q is constant; write $q = e^a$ for some $a \in \mathbb{C}$. It follows that

$$e^{h-a} = f,$$

so $h - a$ is a branch of logarithm of f. Thus every nonvanishing holomorphic function on G has a branch of logarithm. By Characterizations of Simply Connected Regions (7.21), G is simply connected. □

Corollary 9.4 — **Identity Theorem**

Let $G \subseteq \mathbb{C}$ be open and connected, and let $u, v : G \to \mathbb{C}$ be harmonic. If $u = v$ on a nonempty open subset of G, then $u = v$ on G.

Proof: Replacing u by $u - v$ and then considering its real and imaginary parts, it suffices to prove the result when $v = 0$ and u is real-valued. Let

$$A = \operatorname{int}_G(\{z \in G \mid u(z) = 0\}).$$

By assumption, A is nonempty, and it is open in G by definition.

We show that A is also closed in G. Let $z_0 \in \overline{A} \cap G$, and choose $r > 0$ such that $D(z_0, r) \subseteq G$. Since $z_0 \in \overline{A}$, there is a nonempty open disc $B \subseteq A \cap D(z_0, r)$. The disc $D(z_0, r)$ is simply connected, so Existence of Harmonic Conjugates (9.2) gives a harmonic conjugate w of u on $D(z_0, r)$. Hence $F = u + iw$ is holomorphic there.

On B, the function u is identically zero, so Cauchy-Riemann Equations (2.2) implies that $w_x = w_y = 0$. Thus F is constant on B. By Identity Theorem (5.3), F is constant on $D(z_0, r)$. In particular, $u = 0$ on $D(z_0, r)$, so $z_0 \in A$.

Therefore A is nonempty, open, and closed in the connected set G. Hence $A = G$, and so $u = 0$ on G. □

Warning

The cluster-point version of Identity Theorem (5.3) does not hold for harmonic functions. For example, the harmonic functions

$$u(z) = \mathrm{Re}(z) \quad \text{and} \quad v(z) = 0$$

agree on the imaginary axis but are not equal on $\mathbb{C}$.

9.3 Mean-Value and Maximum Principles for Harmonic Functions

Corollary 9.5 **Mean Value Property**

If G is a domain in $\mathbb{C}$, and u is harmonic on G, then for $z_0 \in G$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,\mathrm{d}t$$

for $0 < r < d(z_0, \mathbb{C} \setminus G)$.

Proof: Since the real and imaginary parts of u are harmonic, it suffices to prove the result when u is $\mathbb{R}$-valued. Fix

$$0 < r < d(z_0, \mathbb{C}G),$$

and choose R such that

$$r < R < d(z_0, \mathbb{C}G).$$

Then $D(z_0, R) \subseteq G$. Since the disc $D(z_0, R)$ is simply connected, Existence of Harmonic Conjugates (9.2) gives a harmonic conjugate v of u on $D(z_0, R)$. Thus

$$f = u + iv$$

is holomorphic on $D(z_0, R)$. By Mean Value Property for Holomorphic Functions (4.11),

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

Taking real parts gives

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,\mathrm{d}t,$$

as required. □

Exercise 9.3.1 **Area Mean Value Property**

Under the hypotheses above, show that

$$u(z_0) = \frac{1}{\pi r^2} \int_{\overline{D}(z_0,r)} u(x+iy)\,\mathrm{d}(x,y),$$

where the integral is the two-dimensional Riemann integral over the closed disc $\overline{D}(z_0, r)$.

Corollary 9.6 **Maximum Principle**

If u is a non-constant, real, harmonic function on an open, connected set, then u admits no local maxima.

Proof: Suppose, for a contradiction, that u has a local maximum at z_0. Choose $R > 0$ such that

$$D(z_0, R) \subseteq G \quad \text{and} \quad u(z) \leq u(z_0)$$

for every $z \in D(z_0, R)$. Fix $0 < r < R$. By Mean Value Property (9.5),

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \left(u(z_0) - u(z_0 + re^{it})\right) \mathrm{d}t.$$

The integrand is continuous and nonnegative, so it must vanish identically. Hence

$$u(z_0 + re^{it}) = u(z_0)$$

for every $t \in [0, 2\pi]$. Since this holds for every $0 < r < R$, the function u is constant on $D(z_0, R)$. By Identity Theorem (9.4), u is constant on G, contradicting the hypothesis. □

10 The Argument Principle and Rouché's Theorem

10.1 Counting Zeros and Poles with the Argument Principle

Theorem 10.1 **Argument Principle**

Let $G \subseteq \mathbb{C}$ be a region, and let Γ be a simple contour such that

$$R(\Gamma) \subseteq G \quad \text{and} \quad \operatorname{int}(\Gamma) \subseteq G.$$

Let f be a nonconstant holomorphic function on G with no zeros on $R(\Gamma)$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

is the number of zeros of f in $\operatorname{int}(\Gamma)$, counted with multiplicity.

Proof: Set

$$K = R(\Gamma) \cup \mathrm{int}(\Gamma).$$

This is a compact subset of G. The function f has only finitely many zeros in K: otherwise, compactness would give a limit point of its zeros in K, and Identity Theorem (5.3) would imply that f is identically zero. Since f has no zeros on $R(\Gamma)$, the zeros in K all lie in $\mathrm{int}(\Gamma)$. Denote them by $z_1, \ldots, z_p$, with respective multiplicities $m_1, \ldots, m_p$.

Near z_j, factor

$$f(z) = \left(z - z_j\right)^{m_j} g_j(z),$$

where g_j is holomorphic and $g_j\left(z_j\right) \neq 0$. It follows that

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z - z_j} + \frac{g_{j'}(z)}{g_j(z)}.$$

Thus $\frac{f'}{f}$ has a simple pole at z_j with

$$\mathrm{res}_{z=z_j}\left(\frac{f'}{f}\right) = m_j.$$

Let U be G with every zero of f other than $z_1, \ldots, z_p$ removed. The isolatedness of the zeros implies that U is open, and $K \subseteq U$. Moreover, $\frac{f'}{f}$ is holomorphic on $U \setminus \left\{z_1, \ldots, z_p\right\}$. If $w \notin U$, then $w \notin K$, so the simplicity of Γ gives

$$\mathrm{Ind}_{z=w}(\Gamma) = 0.$$

Therefore Residue Theorem (7.18) applied on U yields

$$\begin{aligned}\int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z &= 2\pi i \sum_{j=1}^{p} \mathrm{Ind}_{z=z_j}(\Gamma)\,\mathrm{res}_{z=z_j}\left(\frac{f'}{f}\right) \\ &= 2\pi i \sum_{j=1}^{p} m_j,\end{aligned}$$

because $z_j \in \mathrm{int}(\Gamma)$ implies $\mathrm{Ind}_{z=z_j}(\Gamma) = 1$. Dividing by $2\pi i$ proves the result. □

Exercise 10.1.1

$$\mathrm{res}_{z_j}\left(\frac{f'}{f}\right) = m_j$$

10.2 Stability of Zeros: Rouché's Theorem

Theorem 10.2 **Rouché's Theorem**

Let $G \subseteq \mathbb{C}$ be a region, and let Γ be a simple contour such that

$$R(\Gamma) \subseteq G \quad \text{and} \quad \operatorname{int}(\Gamma) \subseteq G.$$

Let f and g be holomorphic on G, and suppose that

$$|f(z) - g(z)| < |f(z)|$$

for every $z \in R(\Gamma)$. Then f and g have the same number of zeros in $\operatorname{int}(\Gamma)$, counted with multiplicity.

Remark **Intuition for Rouché's Theorem**

Think of z_0 as a tree, $f(z)$ as the position of a dog, and $g(z)$ as the position of its walker. The inequality

$$|f(z) - g(z)| < |f(z)|$$

says that the leash is shorter than the dog's distance from the tree. As the dog and walker traverse the contour, the leash can never pass across the tree, so their paths have the same winding number about it. The proof below makes this deformation precise.

Proof: For $t \in [0, 1]$, define

$$f_t = (1-t)f + tg = f + t(g - f).$$

If $z \in R(\Gamma)$, then

$$|f_t(z) - f(z)| = t|g(z) - f(z)| < |f(z)|.$$

Hence $f_t(z) \neq 0$ on $R(\Gamma)$ for every $t \in [0, 1]$. By Argument Principle (10.1), the number of zeros of f_t in $\operatorname{int}(\Gamma)$, counted with multiplicity, is

$$N(t) = \frac{1}{2\pi i} \int_\Gamma \frac{f_{t'}(z)}{f_t(z)}\, \mathrm{d}z.$$

The function $(t, z) \mapsto f_{t'} \frac{z}{f_t}(z)$ is continuous on the compact set $[0, 1] \times R(\Gamma)$, so $N : [0, 1] \to \mathbb{C}$ is continuous. Moreover, if f_t is nonconstant, the Argument Principle says that $N(t)$ is its number of zeros in $\operatorname{int}(\Gamma)$. If f_t is constant, it is nonzero and $N(t) = 0$, which is again its number of zeros. Thus $N(t) \in \mathbb{Z}$ for every t. Since $[0, 1]$ is connected, N is constant. Therefore $N(0) = N(1)$, which says that f and g have the same number of zeros in $\operatorname{int}(\Gamma)$, counted with multiplicity. $\square$

Exercise 10.2.1 **Continuous Integer-Valued Functions**

Prove that every continuous function

$$m : [0, 1] \to \mathbb{Z}$$

is constant.

11 Jordan Curves, Retractions, and Planar Separation

11.1 Jordan Curves and Arcs

We follow Maehara's argument for the Jordan Curve Theorem. We will freely use the fact that an open subset of $\mathbb{C}$ is connected if and only if it is path connected.

Definition 11.1 **Jordan Curve**

A *Jordan curve* is the image of an injective continuous function

$$\varphi : \partial\mathbb{D} \to \mathbb{C}.$$

Definition 11.2 **Jordan Arc**

A *Jordan arc* is the image of an injective continuous function

$$\varphi : [0, 1] \to \mathbb{C}.$$

Definition 11.3 **Distance to a Compact Set**

If $K \subseteq \mathbb{C}$ is a nonempty compact set and $z \in \mathbb{C}$, define

$$\operatorname{dist}(z, K) = \inf_{k \in K} |z - k|.$$

For fixed K, this gives a continuous function

$$\operatorname{dist}(\cdot, K) : \mathbb{C} \to [0, \infty).$$

Theorem 11.1 **Tietze Extension Theorem**

Suppose $K \subseteq \mathbb{C}$ is compact and $f : K \to [0, 1]$ is continuous. Then there exists a continuous map $F : \mathbb{C} \to [0, 1]$ such that

$$F(x) = f(x)$$

for every $x \in K$.

Proof: We inductively construct continuous functions $g_n : \mathbb{C} \to [0, 1]$ such that, for $x \in K$,

$$f(x) - \sum_{k=0}^{n} \frac{2^k}{3^{k+1}} g_k(x) \in \left[0, \left(\frac{2}{3}\right)^{n+1}\right].$$

Once this is done, the series

$$\sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}} g_k(x)$$

converges uniformly, since

$$\sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}} g_k(x) \le \sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}} = 1.$$

Its sum therefore defines a continuous function $F : \mathbb{C} \to [0,1]$, and the displayed estimate implies that $F(x) = f(x)$ for every $x \in K$.

Suppose $g_0, ..., g_{n-1}$ have been constructed; for $n = 0$ this assumption is vacuous. Define the continuous function

$$\tilde{f} : K \to \left[0, \left(\frac{2}{3}\right)^{n}\right], \quad \tilde{f}(x) = f(x) - \sum_{k=0}^{n-1} \frac{2^k}{3^{k+1}} g_k(x).$$

Set

$$A = \tilde{f}^{-1}\left(\left[0, \frac{2^n}{3^{n+1}}\right]\right)$$

and

$$B = \tilde{f}^{-1}\left(\left[\frac{2^{n+1}}{3^{n+1}}, \left(\frac{2}{3}\right)^{n}\right]\right).$$

These are disjoint compact sets. If $B = \varnothing$, let g_n be the constant zero function; if $A = \varnothing$, let g_n be the constant one function. Otherwise,

$$\delta = \inf_{a \in A} \operatorname{dist}(a, B) > 0.$$

Define $g_n(z) = h_\delta(\operatorname{dist}(z, B))$, where $h_\delta : [0, \infty) \to [0,1]$ is continuous and

$$h_\delta(t) = \begin{cases} 1 - \delta^{-1} t & t \in [0, \delta] \\ 0 & t > \delta. \end{cases}$$

Then $g_n(z) = 1$ on B and $g_n(z) = 0$ on A. Consequently, for $x \in K$,

$$f(x) - \sum_{k=0}^{n} \frac{2^k}{3^{k+1}} g_k(x) = \tilde{f}(x) - \frac{2^n}{3^{n+1}} g_n(x) \in \left[0, \left(\frac{2}{3}\right)^{n+1}\right].$$

This completes the induction. □

11.2 Retractions and the Brouwer Fixed Point Theorem

Definition 11.4 **Retract**

If $K_1 \subseteq K_2 \subseteq \mathbb{C}$, a *retract* of K_2 onto K_1 is a continuous function

$$r : K_2 \to K_1$$

whose restriction to K_1 is the identity.

Corollary 11.2 **Retraction onto a Jordan Arc**

If $A \subseteq \mathbb{C}$ is a Jordan arc, then there is a retract from $\mathbb{C}$ onto A.

Proof: Let $\gamma : [0,1] \to \mathbb{C}$ be continuous and injective with $A = \gamma([0,1])$. The set A is compact, and

$$\gamma^{-1} : A \to [0,1]$$

is continuous. By Tietze Extension Theorem (11.1), there is a continuous $F : \mathbb{C} \to [0,1]$ such that $F(z) = \gamma^{-1}(z)$ for $z \in A$. Then

$$r = \gamma \circ F$$

is a retract from $\mathbb{C}$ onto A. □

Theorem 11.3 **Non-Retraction Theorem**

There is no retract from $\overline{\mathbb{D}}$ onto $\partial\mathbb{D}$.

Proof: Suppose $r : \overline{\mathbb{D}} \to \partial\mathbb{D}$ were a retract. Define

$$\gamma_0(z) = z, \quad \gamma_1(z) = 1$$

on $\partial\mathbb{D}$, and let

$$h : [0,1] \times \partial\mathbb{D} \to \overline{\mathbb{D}}, \quad h(t,z) = (1-t)z + t.$$

Then

$$k(t,z) = r(h(t,z))$$

is a continuous homotopy in $\partial\mathbb{D}$ from γ_0 to γ_1.

For $s \in [0,1]$, let $\gamma_s : \partial\mathbb{D} \to \mathbb{C}$ be the closed curve

$$\gamma_s(z) = k(s,z).$$

Uniform continuity gives $\delta > 0$ such that, whenever $|s_1 - s_2| < \delta$,

$$\left|\gamma_{s_1}(z) - \gamma_{s_2}(z)\right| < 1$$

for every $z \in \partial\mathbb{D}$. Hence

$$\left| \frac{\gamma_{s_2}(z)}{\gamma_{s_1}(z)} - 1 \right| < 1,$$

so $\frac{\gamma_{s_2}}{\gamma_{s_1}}$ takes values in the right half-plane. Choosing a branch of the logarithm there gives a continuous $\varphi : \partial\mathbb{D} \to \mathbb{C}$ such that

$$\gamma_{s_2}(z) = e^{\varphi(z)}\gamma_{s_1}(z).$$

It follows that

$$\mathrm{ind}_{\gamma_{s_1}}(0) = \mathrm{ind}_{\gamma_{s_2}}(0).$$

Thus the continuous integer-valued function

$$[0,1] \to s \to \mathrm{ind}_{\gamma_s}(0)$$

is constant. This contradicts

$$\mathrm{ind}_{\gamma_0}(0) = 1 \quad \text{and} \quad \mathrm{ind}_{\gamma_1}(0) = 0.$$

□

Exercise 11.2.1 **Uniformity in a Continuous Family of Curves**

Suppose $h : [0,1] \times \partial\mathbb{D} \to \mathbb{C}$ is continuous. Show that for each $\varepsilon > 0$ there is $\delta > 0$ such that, if $s, t \in [0,1]$ and $|s - t| < \delta$, then

$$\sup_{z \in \partial\mathbb{D}} |h(s,z) - h(t,z)| < \varepsilon.$$

Corollary 11.4 **Brouwer Fixed Point Theorem in Two Dimensions**

Every continuous function

$$f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$$

has a fixed point.

Proof: Let

$$\Delta = \{(z,z) \mid |z| \le 1\} \subseteq \overline{\mathbb{D}}^2$$

be the diagonal. Define

$$r_0 : \overline{\mathbb{D}}^2 \setminus \Delta \to \partial\mathbb{D}$$

by letting $r_0(z_1, z_2)$ be the unique point $z \in \partial\mathbb{D}$ such that z_2 lies on the line segment $[z_1, z]$. This map is continuous.

If $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ had no fixed point, then

$$r(z) = r_0(f(z), z)$$

would define a continuous map $r : \overline{\mathbb{D}} \to \partial\mathbb{D}$. Moreover, $r(z) = z$ for $z \in \partial\mathbb{D}$, so r would be a retract, contradicting Non-Retraction Theorem (11.3). □

Remark

More generally, if $K \subseteq \mathbb{C}$ is homeomorphic to $\overline{\mathbb{D}}$, then every continuous map $K \to K$ has a fixed point. In particular, every rectangle $[a, b] \times [c, d]$ with $a < b$ and $c < d$ is homeomorphic to $\overline{\mathbb{D}}$.

11.3 Planar Separation and the Jordan Curve Theorem

Lemma 11.5 **Boundary of a Bounded Complementary Component**

Let J be a Jordan curve. Every bounded connected component of $\mathbb{C} \setminus J$ has J as its boundary.

Proof: Let U be a bounded connected component of $\mathbb{C} \setminus J$. Then U is open and disjoint from the other components, so

$$\partial U = \overline{U} \setminus U \subseteq J.$$

Suppose the inclusion were strict. Then there would be a Jordan arc $A \subsetneq J$ such that $\partial U \subseteq A$. Fix $o \in U$, and let D be a closed disc centered at o whose interior contains J. By Retraction onto a Jordan Arc (11.2), there is a retract $r : D \to A$.

Define $q : D \to D \setminus \{o\}$ by

$$q(z) = \begin{cases} r(z) z \in \overline{U} \\ z \quad\; z \in U^c. \end{cases}$$

Since $r(z) = z$ for $z \in \partial U$, the map q is continuous. Let $p : D \setminus \{o\} \to \partial D$ be radial projection from o. Then $p \circ q$ is a retract from D onto ∂D, contradicting Non-Retraction Theorem (11.3) after identifying D with $\overline{\mathbb{D}}$. □

The next lemma is the topological fact underlying the game of Hex.

Lemma 11.6 **Crossing Lemma**

Suppose $a < b$ and $c < d$. Let

$$h, k : [-1, 1] \to [a, b] \times [c, d]$$

be continuous, with

$$h(-1) \in \{a\} \times [c, d], h(1) \in \{b\} \times [c, d],$$
$$k(-1) \in [a, b] \times \{c\}, k(1) \in [a, b] \times \{d\}.$$

Then there are $s, t \in [-1, 1]$ such that $h(s) = k(t)$.

Proof: Write

$$h(s) = (h_1(s), h_2(s)) \quad \text{and} \quad k(t) = (k_1(t), k_2(t)).$$

Suppose $h(s) \neq k(t)$ for every $s, t \in [-1, 1]$, and define

$$N(s, t) = \max(|h_1(s) - k_1(t)|, |h_2(s) - k_2(t)|).$$

Then the map $F : [-1, 1]^2 \to [-1, 1]^2$ given by

$$F(s, t) = \left(\frac{k_1(t) - h_1(s)}{N(s, t)}, \frac{h_2(s) - k_2(t)}{N(s, t)} \right)$$

is continuous and takes values on the boundary of $[-1, 1]^2$.

If $F(s, t) = (s, t)$, then either $|s| = 1$ or $|t| = 1$. If $|s| = 1$, the first coordinate of $F(s, t)$ has the opposite sign from s; if $|t| = 1$, the second coordinate has the opposite sign from t. Thus F has no fixed point, contradicting Brouwer Fixed Point Theorem in Two Dimensions (11.4). □

Theorem 11.7 **Jordan Curve Theorem**

If $J \subseteq \mathbb{C}$ is a Jordan curve, then $\mathbb{C} \setminus J$ has exactly one bounded connected component.

Proof: Since J is compact, choose $a, b \in J$ such that

$$|a - b| = \operatorname{diam}(J) = \sup_{z_1, z_2 \in J} |z_1 - z_2|.$$

After rotating, translating, and scaling, assume

$$a = (-1, 0) \quad \text{and} \quad b = (1, 0).$$

Then

$$J \subseteq [-1, 1] \times [-2, 2],$$

and J meets the boundary of this rectangle exactly at a and b. Decompose J into Jordan arcs A and B with

$$A \cap B = \{a, b\}.$$

By Crossing Lemma (11.6), both A and B meet the vertical segment $[-2i, 2i]$. Let m be the point of $[-2i, 2i] \cap J$ with largest imaginary part and, after relabeling if necessary, suppose $m \in A$. Let n be the point of $[-2i, 2i] \cap A$ with smallest imaginary part, and let $A_0 \subseteq A$ be the Jordan arc from n to m; take $A_0 = \varnothing$ if $n = m$.

The arc B must meet $[-2i, n]$. Otherwise, the curve

$$[-2i, n] \cup A_0 \cup [m, 2i]$$

and the arc B would contradict Crossing Lemma (11.6). Let p and q be, respectively, the points of $[-2i, n] \cap B$ with largest and smallest imaginary parts. Let $B_0 \subseteq B$ be the Jordan arc from p to q, taking $B_0 = \varnothing$ if $p = q$. Choose an interior point z_0 of $[p, n]$, and let U be the connected component of $\mathbb{C} \setminus J$ containing z_0.

We claim that U is bounded. Otherwise, because U is path connected, there would be a Jordan arc $C \subseteq U$ from z_0 to some point w on the boundary of $[-1, 1] \times [-2, 2]$. If w lies on the lower half of this boundary away from a and b, join $-2i$ to w along the boundary by a path γ. Then

$$\gamma \cup C \cup [z_0, n] \cup A_0 \cup [m, 2i]$$

is a curve from $-2i$ to $2i$ that misses B, contradicting Crossing Lemma (11.6). If w lies on the upper half, the analogous construction gives a curve from $2i$ to $-2i$ that misses A, again a contradiction. Therefore U is a bounded connected component of $\mathbb{C} \setminus J$.

Suppose V were another bounded connected component. Since J lies in the rectangle, so does V. Consider

$$\gamma_0 = [-2i, q] \cup B_0 \cup [p, n] \cup A_0 \cup [m, 2i].$$

This curve misses V and contains neither a nor b. Hence there is $\delta > 0$ such that the δ-balls about a and b miss γ_0. By Boundary of a Bounded Complementary Component (11.5), choose $a_0, b_0 \in V$ such that

$$|a - a_0| < \delta \quad \text{and} \quad |b - b_0| < \delta.$$

Since V is path connected, take a curve $\gamma \subseteq V$ joining a_0 to b_0. Then

$$[a, a_0] \cup \gamma \cup [b_0, b]$$

joins the left side of the rectangle to the right side while missing γ_0, contradicting Crossing Lemma (11.6). Thus U is the unique bounded connected component of $\mathbb{C} \setminus J$. $\square$