Contents

1 Topological Spaces and Continuous Maps

1.1 Mathematical Spaces

Recall

Given a real inner product $\langle , \rangle$ on a real vector space V, we can define a norm on V by $\|x\| = \sqrt{\langle x, x \rangle}$.
Given a norm on a real vector space V, we can define a metric on V by $d(x, y) = \|x - y\|$.
Given a metric d on a set X, we can define open sets in X as follows:

Definition 1.1.1

Given $a \in X$ and $r > 0$, $B(a, r) = B_X(a, r) = \{x \in X \mid d(a, x) < r\}$, then for $A \subseteq X$, we say that A is open (in X) when $\forall a \in A, \exists r > 0$ such that $B(a, r) \subseteq A$.

Note

Equivalently, A is open (in X) when $\forall a \in A, \exists b \in X, \exists r > 0$ such that $a \in B(b, r) \subseteq A$.

Remark

The set of open sets in the metric space X is called the metric topology on X. Open sets in a metric space satisfy the following:

1. $\varnothing$ and X are open.
2. The union of any set of open sets in X is open (in X).
3. The intersection of any finite set of open sets in X is open (in X).

Notation

1. For a set S of sets, the union of S is $\cup S := \{x | \exists A \in S, x \in A\}$.
2. For $S \neq \varnothing$, the intersection of S is $\cap S := \{x | \forall A \in S, x \in A\}$.
3. In the case that $S = \{A_k \mid k \in K\}$ where K is a set (called the index set), then $\cup S = \bigcup_{k \in K} A_k \cup S = \bigcup_{k \in K} A_k$.

Definition 1.1.2

Let X be a set. A topology on X is a set $\mathscr{T}$ of subsets of X such that

1. $\varnothing \in \mathscr{T}$ and $X \in \mathscr{T}$
2. $\mathscr{T}$ is closed under arbitrary unions
3. $\mathscr{T}$ is closed under finite intersections

Remark

By induction, to verify that (3) holds, it suffices to show that
$A, B \in \mathscr{T} \Rightarrow A \cap B \in \mathscr{T}$.

Definition 1.1.3

When $\mathscr{T}$ is a topology on X, the elements of $\mathscr{T}$ are called the open sets in X. The complements (in X) are called the closed sets in X.

Example

X set. $\mathscr{T} = \{\varnothing, X\}$ is a topology. Moreover, it's called the trivial topology.
Also, $P = P(X) = \{A | A \subseteq X\}$ is a topology on X. It's called the discrete topology.

Recall

When $f : X \to Y$ is a map between metric spaces, f is continuous ⇔ f has the property that $f^{-1}(V)$ is open in X for every open set V in Y.

Definition 1.1.4

For a map $f : X \to Y$ between topological spaces, we say that f is continuous when $f^{-1}(V)$ is open in X for every open V in Y.

Example

Recall that if $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$ is an elementary function, then f is continuous.

Definition 1.1.5

A topological space is a set X together with a topology $\mathscr{T}$.

Definition 1.1.6

Let $\mathscr{S}, \mathscr{T}$ be topologies on X with $\mathscr{S} \subseteq \mathscr{T}$, we say that $\mathscr{S}$ is coarser than $\mathscr{T}$ and that $\mathscr{T}$ is finer than $\mathscr{S}$. When $\mathscr{S} \subsetneq \mathscr{T}$, we say that $\mathscr{S}$ is strictly coarser than $\mathscr{T}$ and that $\mathscr{T}$ is strictly finer than $\mathscr{S}$.

Example

Consider $X = \varnothing$. $T \subseteq P(X) = \{\varnothing\} \Rightarrow T = \varnothing$ or $T = \{\varnothing\}$. But only $T = \{\varnothing\}$ is a topology since $X = \varnothing \notin T$.

Example

Consider $X = \{a\}$. $\{\varnothing, \{a\}\} = P(X) \supseteq T \in \{\varnothing, \{\varnothing\}, \{\{a\}\}, \{\varnothing, \{a\}\}\}$. But $\varnothing \notin \varnothing$, $\{a\} \notin \{\varnothing\}$ and $\varnothing \notin \{\{a\}\}$. So $T = \{\varnothing, \{a\}\}$ is the only remaining candidate. We check that it is closed under arbitrary unions and finite intersections to conclude that it is a topology on X.

Exercise 1.1.1

Find all topologies on $X = \{a, b\}$ and $X = \{a, b, c\}$.

Solution:
Let $X = \{a, b\}$. Then $T \subseteq P(\{a, b\})$ such that $\varnothing \in T$ and $\{a, b\} \in T$. So
$T \in P = \{\{\varnothing, \{a, b\}\}, \{\varnothing, \{a, b\}, \{a\}\}, \{\varnothing, \{a, b\}, \{b\}\}, \{\varnothing, \{a, b\}, \{a\}, \{b\}\}\}$.
Note that each $T \in P$ is closed under arbitrary unions and finite intersections, so each one is a topology on X.
Similarly,
$S_T = \{\{\varnothing, \{a, b, c\}\}, \{\varnothing, \{a, b, c\}, \{a\}\}, \{\varnothing, \{a, b, c\}, \{b\}\}, \{\varnothing, \{a, b, c\}, \{c\}\}, \dots\}$
for $X = \{a, b, c\}$.

Definition 1.1.7

Let X be a topological space and let $A \subseteq X$.

1. The interior of A (in X), denoted by A^0, or by $\operatorname{int}(A)$, or by $\operatorname{int}_X(A)$, is the union (of the set) of all open sets in X, which are contained in A.
2. The closure of A (in X), denoted by $\overline{A}$, or by $Cl(A)$, or by $Cl_X(A)$, is the intersection of (the set of) all the closed sets in X, which contain A
3. The boundary of A (in X), denoted by δA, or by $\delta_X A$, or by $Bd(A)$, or by $Bd_X(A)$, given by $\delta A = \overline{A} \setminus A^0 = \left\{x \mid x \in \overline{A}, x \notin A^0\right\}$.

Note

The set of closed sets in a topological space is closed under arbitrary intersection and under finite unions. So, in particular, $\varnothing$ and X are both closed.

Theorem 1.1

Let X be a topological space and let $A \subseteq X$. Then

1. A^0 is open in X, and it is the largest open set in X, which is contained in A. Also, $\overline{A}$ is closed in X, and $\overline{A}$ is the smallest closed set in X, which contains A.
2. A is open $\Leftrightarrow A = A^0$. Moreover, A is closed $\Leftrightarrow A = \overline{A}$
3. $\left(A^0\right)^0 = A^0$ and $\overline{\overline{A}} = \overline{A}$

Proof:

1. A^0 is a union of open sets in X, so it is open in X. Moreover, there cannot exist open B in X such that $A^0 \subsetneq B \subseteq A$, since A^0 is the union of all open sets in X contained in A, so $B \subseteq A^0$.
   Similarly, $X \setminus \overline{A}$ is a union of open sets in X, so it is open in X. Hence, $\overline{A}$ is open. Moreover, there cannot exist closed B in X such that $A \subseteq B \subsetneq \overline{A}$ because $\overline{A}$ is the intersection of all closed sets in X containing A, so $\overline{A} \subseteq B$.
2. A open $\Rightarrow A^0 = A$ since A^0 is the largest open set contained in A and $A \subseteq A$.
   $A^0 = A \Rightarrow A$ open since A^0 is open from part (1).
   A closed $\Rightarrow \overline{A} = A$ since $\overline{A}$ is the smallest closed set containing A and $A \subseteq A$.
   $\overline{A} = A \Rightarrow A$ closed since $\overline{A}$ is closed.
3. $(A^0)^0 = A^0$ since A^0 is open and since $(A^0)^0$ is the largest open set contained in A^0. Moreover, $\overline{\overline{A}} = \overline{A}$ since $\overline{A}$ is closed and since $\overline{\overline{A}}$ is the smallest closed set containing $\overline{A}$.

□

Definition 1.1.8

Let X be a topological space and let $A \subseteq X$ with $a \in A$.
1. We say that a is an interior point of A when $a \in A$ and there exists an open set U in X with $a \in U \subseteq A$.
2. We say that a is a limit point of A when for every open set U in X with $a \in U$, we have $(U \setminus \{a\}) \cap A \neq \varnothing$.
   The set of limit points of A in X is denoted by A^1, or by $Lim(A)$.
3. We say that a is a boundary point of A (in X) when for every open set U in X with $a \in U$, we have $U \cap A \neq \varnothing$ and $U \cap A^c \neq \varnothing$ (where $A^c = X \setminus A$).

Theorem 1.2

Let X be a topological space and let $A \subseteq X$.
1. A^0 is equal to the set of all interior points of A in X.
2. For $a \in X$, $a \in A^1 \Leftrightarrow a \in \overline{A \setminus \{a\}}$.
3. A is closed $\Leftrightarrow A^1 \subseteq A$.
4. $\overline{A} = A \cup A^1$.
5. $\overline{A}$ is the disjoint union $\overline{A} = A^0 \cup \delta A$.
6. δA is equal to the set of boundary points of A in X.

Proof:
1. Let A_I denote the set of interior points in A. $a \in A^0 \Rightarrow a \in U \subseteq A$, where U is open. So $a \in A_I$. If $a \in A_I$, then there exists open U such that $a \in U \subseteq A$. So $a \in A^0$. Hence, $A^0 = A_I$.
2. Let $a \in X$. $a \in A^1$. Let C be a closed set such that $A \setminus \{a\} \subseteq C$.
3.

□

1.2 Topological bases

Theorem 1.3

Let X be a topological space. The intersection of any set of topologies on X is also a topology on X.

Proof:

Let R be a set of topologies on X. When $R = \varnothing$, since the elements of R lie in the power set $P(X)$, we take $\cap R$ to be the set $P(X)$, which is the discrete topology on X.
Suppose $R \neq \varnothing$. $\forall A \in R, \varnothing \in A$ since A is a topology, so $\varnothing \in \cap R$. Similarly, $\forall A \in R, X \in A$ since A is a topology on X, so $X \in \cap R$.
Now, let $S \subseteq \cap R$. For $A \in R$, since $S \subseteq \cap R$ and $A \in R$, then $S \subseteq A$. But A is a topology so it is closed under arbitrary unions. Therefore, $\cup S \in A$. But $A \in R$ is arbitrary, so $\cup S \in \cap R$. Hence, $\cap R$ is closed under arbitrary unions.
Finally, let $T_1, T_2 \in \cap R$. For $A \in R$, since $T_1 \in \cap R$ and $A \in R$, then $T_1 \in A$. Similarly, $T_2 \in A$. But A is a topology, so $T_1 \cap T_2 \in A$. However, $A \in R$ was arbitrary, so $T_1 \cap T_2 \in \cap R$. Hence, $\cap R$ is closed under finite intersection. □

Note

Given a set X, the union of any set of topologies is not a topology on X in general.

Corollary 1.4

When X is a set and $\mathscr{S}$ is any set of subsets of X (that is, $\mathscr{S} \subseteq P(X)$), there exists a unique smallest (coarsest) topology $\mathscr{T}$ on X which contains $\mathscr{S}$. In particular, $\mathscr{T}$ is the intersection of (the set of) all topologies on X containing $\mathscr{S}$.

Note

This topology $\mathscr{T}$ is called the topology on X generated by $\mathscr{S}$.

Definition 1.2.1

Let X be a set. A basis of sets on X is a set $\mathscr{B}$ of subsets of X (so $\mathscr{B} \subseteq P(X)$) such that:

1. $\mathscr{B}$ covers X (that is, $\cup \mathscr{B} = X$)
2. $\forall C, D \in \mathscr{B}, \forall a \in C \cap D, \exists B \in \mathscr{B}, a \in B \subseteq C \cap D$.

Note

When $\mathscr{B}$ is a basis of sets in X and $\mathscr{T}$ is the topology on X generated by $\mathscr{B}$, we say that $\mathscr{B}$ is a basis for $\mathscr{T}$.
The elements in $\mathscr{B}$ are called the basic open sets in X.

Theorem 1.5 **Characterization of Open Sets in terms of Basic Open Sets**

Let X be a topological space, $\mathscr{B}$ be a basis for the topology on X.

1. For $A \subseteq X$, A is open in X if and only if A has the property that $\forall a \in A, \exists B \in \mathscr{B}, a \in B \subseteq A$.
2. The open sets in X are the unions of (the set of) elements in $\mathscr{B}$.

Proof:

Let $\mathcal{T}$ be the topology on X generated by the basis $\mathscr{B}$. Let $\mathcal{S}$ be the set of all sets $A \subseteq X$ with the property $\forall a \in A, \exists B \in \mathscr{B}, a \in B \subseteq A$ $(**)$ and let $\mathcal{R}$ be the set of (arbitrary) unions of (sets of) elements in $\mathscr{B}$.
Recall that $\mathcal{T}$ is the intersection of the set of all topologies on X which contain $\mathscr{B}$. Note that $\mathscr{S}$ contains $\mathscr{B}$, since

$$\begin{aligned} A \in \mathscr{B} &\Rightarrow \forall C, D \subseteq X, \forall a \in C \cap D, \exists B \in \mathscr{B}, a \in B \subseteq C \cap D \\ &\Rightarrow \forall A \subseteq X, \forall a \in A, \exists B \in \mathscr{B}, a \in B \subseteq A \\ &\Rightarrow A \in \mathscr{S} \end{aligned}$$

Now, let us show that $\mathscr{S}$ is a topology on X.
We have $\varnothing \in \mathscr{S}$ vacuously and $X \in \mathscr{S}$ because $\mathscr{B}$ covers X (given $a \in X$, we can choose $B \in \mathscr{B}$ with $a \in B$). When $U_k \in S$ for every $k \in K$ (where K is any index set), we need to show that $\cup_{k \in K} U_k \in \mathscr{S}$.
Let $a \in \cup_{k \in K} U_k$. Choose $l \in K$ so that $a \in U_l$. Since $U_l \in \mathscr{S}$ (so U_l satisfies $**$), we can choose $B \in \mathscr{B}$ so that $a \in B \subseteq U_l$. Since $U_l \subseteq \cup_{k \in K} U_k$, we have $a \in B \subseteq \cup_{k \in K} U_k$. Thus $\cup_{k \in K} U_k$ satisfies $**$, hence $\cup_{k \in K} U_k \in \mathscr{S}$, as required.
Suppose $U, V \in \mathscr{S}$ (so U and V satisfy $**$).
Let $a \in U \cap V$. Since $U \in \mathscr{S}$, we can choose $C \in \mathscr{B}$ with $a \in C \subseteq U$.
Since $V \in \mathscr{S}$, then we can choose $D \in \mathscr{B}$ with $a \in D \subseteq V$. Since $\mathscr{B}$ is a basis, $C, D \in \mathscr{B}$ and $a \in C \cap D$, we can choose $B \in \mathscr{B}$ with $a \in \mathscr{B} \subseteq C \cap D$.
Then we have $a \in \mathscr{B} \subseteq C \cap D \subseteq U \cup V$. Thus $U \cap V$ satisfies $**$, so that $U \cap V \in \mathscr{S}$, as required. Thus $\mathcal{S}$ is a topology on X containing $\mathscr{B}$, hence $\mathscr{T} \subseteq \mathscr{S}$.
Let us show that $\mathscr{S} \subseteq \mathcal{R}$. Let $U \in \mathscr{S}$ (so U satisfies $**$). For each $a \in U$, choose $B_a \in \mathscr{B}$ with $a \in B_a \subseteq U$. Then we have $U = \cup_{a \in U} B_a \in \mathcal{R}$. Thus $\mathscr{S} \subseteq \mathscr{R}$.
Finally, note that $\mathscr{R} \subseteq \mathscr{T}$ because if $U = \cup_{k \in K} B_k$ with $B_k \in \mathscr{B}$, then each $B_k \in \mathscr{T}$ and $\mathscr{T}$ is a topology, so $U = \cup_{k \in K} B_k \in \mathscr{T}$.
Therefore, we have that $\mathcal{T} \subseteq \mathcal{S} \subseteq \mathcal{R} \subseteq \mathcal{T}$, so $\mathcal{R} = \mathcal{S} = \mathcal{T}$. But $\mathcal{S} = \mathcal{T}$ is equivalent to (1) and $\mathcal{R} = \mathcal{T}$ is equivalent to (2). $\square$

Theorem 1.6 **Characterization of a Basis in terms of Open Sets**

Let X be a topological space with topology $\mathcal{T}$ and let $\mathscr{B} \subseteq \mathcal{T}$. Then $\mathscr{B}$ is a basis for $\mathcal{T}$ iff $\forall U \in \mathcal{T}, \forall a \in U, \exists B \in \mathscr{B}, a \in B \subseteq U$ $(*)$.

Proof: If $\mathscr{B}$ is a basis for $\mathcal{T}$, then $(*)$ holds by part (1) of the previous theorem.
Suppose that $(*)$ holds. Let's show that $\mathscr{B}$ is a basis of sets in X.
Note that $\mathscr{B}$ covers X since, taking $U = X$ in $(*)$, we have
$\forall a \in X, \exists B \in \mathscr{B}, a \in B \ (\subseteq X)$. Also note that given $C, D \in \mathscr{B}$ and $a \in C \cap D$, then by taking $U = C \cap D$ in $(*)$ (noting that $C, D \in \mathscr{B} \subseteq \mathcal{T}$, so that $U = C \cap D \in \mathcal{T}$), we can choose $B \in \mathscr{B}$ with $a \in B \subseteq C \cap D$. Thus, $\mathscr{B}$ is a basis of sets in X.
It remains to show that $\mathcal{T}$ is the topology generated by $\mathscr{B}$.
Let $\mathscr{S}$ be the topology generated by $\mathscr{B}$. By part (2) of the previous theorem, $\mathscr{S}$ is the set of all unions of elements in $\mathscr{B}$. Also, $\mathscr{S}$ is the smallest topology which contains $\mathscr{B}$. Since $\mathscr{B} \subseteq \mathcal{T}$, and $\mathcal{T}$ is a topology, $\mathscr{S} \subseteq \mathcal{T}$.
Also, we have $\mathcal{T} \subseteq \mathscr{S}$ because given $U \in \mathcal{T}$, by property $(*)$, for each $a \in U$, we can choose $B_a \in \mathscr{B}$ with $a \in B_a \subseteq U$, and then we have $\cup_{a \in U} B_a \in \mathcal{S}$ since it is a union of elements in $\mathscr{B}$. □

Example

When X is a metric space, the set $\mathscr{B}$ of all open balls in X is a basis for the metric topology on X.

Remark

We can use a basis for testing for various topological properties:

1. When X is a topological space, $\mathscr{B}$ is a basis for the topology on X, and $S \subseteq X$ and $a \in X$, then $a \in A^0 \Leftrightarrow \exists B \in \mathscr{B}$ with $a \in B \subseteq A$
2. $a \in \overline{A} \Leftrightarrow \forall B \in \mathscr{B}$ with $a \in B, B \cap A \neq \varnothing$
3. $a \in A^1 \Leftrightarrow \forall B \in \mathscr{B}$ with $a \in B, (B \setminus \{a\}) \cap A \neq \varnothing$
4. $a \in \delta A \Leftrightarrow \forall B \in \mathscr{B}$ with $a \in B, B \cap A \neq \varnothing \wedge B \cap (X \setminus A) \neq \varnothing$

Definition 1.2.2

A topological space X is called Hausdorff when it has the property that for all $a, b \in X$, with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$.

Example

Metric spaces are Hausdorff.

Definition 1.2.3 **Subspace Topology**

Let Y be a topological space with topology $\mathscr{S}$, and $X \subseteq Y$ be a subset. Let $\mathcal{T} = \{V \cap X \mid V \in \mathscr{S}\}$. Then $\mathcal{T}$ is a topology on X. Indeed, $\varnothing \in \mathscr{S}$, so $\varnothing \cap X = \varnothing \in \mathcal{T}$ and $Y \in \mathscr{S}$, so $Y \cap X = X \in \mathcal{T}$. If K is any index set and $U_k \in \mathcal{T}$ for each $k \in K$, then for each $k \in K$, we can choose $V_k \in \mathscr{S}$ such that $U_k = V_k \cap X$, and then we have

$$\bigcup_{k\in K} U_k = \bigcup_{k\in K} (V_k \cap X) = \left(\bigcup_{k\in K} V_k\right) \cap X \in \mathcal{T}$$

since

$$\bigcup_{k\in K} V_k \in \mathscr{S}$$

Similarly, when K is finite and $U_k \in \mathcal{T}$ for each $k \in K$, we have $\bigcap_{k\in K} U_k \in \mathcal{T}$. The topology $\mathcal{T}$ on X is called the subspace topology on X (inherited from the topology on Y).

Theorem 1.7

Let Y be topological space, $\mathscr{C}$ be a basis for the topology on Y and let $X \subseteq Y$ be a subset. Then the set

$$\mathscr{B} = \{C \cap X \mid C \in \mathscr{C}\}$$

is a basis for the subspace topology on X.

Proof:

First, we prove that $\mathscr{B}$ is a basis of sets on X. Observe that

$$\cup \mathscr{B} = \bigcup_{k\in K} \mathscr{B}_k = \bigcup_{k\in K} (C_k \cap X) = \left(\bigcup_{k\in K} C_k\right) \cap X = Y \cap X = X$$

since $\mathscr{S}$ is a basis on Y. Moreover, for $C, D \in \mathscr{B}$, let $a \in C \cap D$ be arbitrary. Then $\exists F \in \mathscr{C}, C = F \cap X$ and $\exists G \in \mathscr{C}, D = G \cap X$. So $a \in (F \cap G) \cap X \subseteq F \cap G$. Since $\mathscr{C}$ is a basis of sets on Y (since it is a basis for the topology on Y), then $\exists C_0 \in \mathscr{C}$ such that $a \in C_0 \subseteq F \cap G$. But $a \in X$, so $a \in C_0 \cap X \subseteq (F \cap G) \cap X = C \cap D$. But $C_0 \cap X \in \mathscr{B}$, and since $a \in C \cap D$ was arbitrary, then $\mathscr{B}$ satisfies property 2 of Definition 1.2.1. So $\mathscr{B}$ is a basis of sets.
Now, we argue that $\mathscr{B}$ generates X. Let $\mathcal{S}$ be the subspace topology on X, $\mathcal{R}$ be the set of all topologies on X containing $\mathscr{B}$. We must show that $\cap \mathcal{R} = \mathcal{S}$.
$\cap \mathcal{R} = \{A \mid \forall B \in \mathcal{R}, A \in B\}$. $\mathcal{S} = \{C \cap X \mid C \in \mathscr{C}\}$. $\square$

Theorem 1.8

Let Z be a topological space. Let $Y \subseteq Z$ be a subspace (meaning that Y uses the subspace topology) and $X \subseteq Y$ be a subset. Then the subspace topology on X inherited from Y is equal to the subspace topology on X inherited from Z.

Proof: Exercise. □

Theorem 1.9

Let Y be a metric space (using the metric space topology) and let $X \subseteq Y$. Then the subspace topology on X (inherited from the topology on Y) is equal to the metric topology on X, using the metric on X obtained by restricting the metric on Y.

Proof: Exercise. □

1.3 Continuous Maps

Definition 1.3.1 **Continuity at a Point**

Let X, Y be topological spaces. For $f : X \to Y$ and $a \in X$, we say that f is continuous at a when for every open sets $V \in Y$ with $f(a) \in V$, there exists an open set $U \in X$ with $a \in U \subseteq f^{-1}(V)$. We say that f is continuous (in or on X) when for every open set $V \in Y$, $f^{-1}(V)$ is open in X.

Definition 1.3.2 **Homeomorphism**

A homeomorphism from X to Y is a bijective map $f : X \to Y$ such that both f and its inverse $f^{-1} : Y \to X$ are continuous.

Note

We say that X and Y are *homeomorphic*, and we write $X \cong Y$ when there exists a homeomorphism $f : X \to Y$ (and we remark that then $f^{-1} : Y \to X$ is also a homeomorphism).

Theorem 1.10

Constant maps and inclusion maps are continuous.

Proof: For $f : X \to Y$ given by $f(x) = c \in Y$ for all $x \in X$, when V is open in Y, $f^{-1}(V) = \begin{cases} X \text{ if } c \in V \\ \varnothing \text{ if } c \notin V \end{cases}$. When $X \subseteq Y$ is is a subspace and $f : X \to Y$ is given by $f(x) = x$ for all $x \in X$, when V is open in Y,

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} = \{x \in X \mid x \in V\} = V \cap X$$

which is open in X (when X is using the subspace topology). □

Remark

When Y is a topological space and $X \subseteq Y$ is a subset, we shall assume, unless indicated otherwise, that X uses the subspace topology.

Theorem 1.11 **Equivalent Definitions of Continuity**

Let $f : X \to Y$ be a map between topological spaces.

1. f is continuous (on X) $\Leftrightarrow$ f is continuous at every $a \in X$.
2. f is continuous (on X) $\Leftrightarrow$ for every closed set K in Y, $f^{-1}(K)$ is closed in X.
3. if $\mathscr{C}$ is a basis for the topology on Y, then f is continuous $\Leftrightarrow$ for every $C \in \mathscr{C}$ $f^{-1}(C)$ is open in X.

Proof:

1. $\Longrightarrow$ Suppose f is continuous on X. Let $a \in X$. Let V be an open set in Y with $f(a) \in V$. Let $U = f^{-1}(V)$. Then $f^{-1}(V)$ is open since f is continuous and $a \in U \subseteq f^{-1}(V)$.
   $\Longleftarrow$ Suppose, conversely, that f is continuous at every $a \in X$. Let V be an open set in Y. For each $a \in f^{-1}(V)$, since f is continuous at a with $f(a) \in V$, we can choose an open set $U_a \in X$ with $a \in U_a \subseteq f^{-1}(V)$. Then $f^{-1}(V) = \cup_{a \in f^{-1}(V)} U_a$, which is open in X since it is a union of open sets in X.

The rest is an Exercise $\square$

Theorem 1.12 **Composites of Continuous Functions are Continuous**

Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between topological spaces. Then the composite map $h = g \circ f : X \to Z$ is continuous.

Proof: Show that $h^{-1}(w) = f^{-1}(g^{-1}(w))$. $\square$

Remark

Homeomorphism of topological spaces behaves like an equivalence relation on he class of topological spaces for topological spaces X, Y, Z.
$X \cong X$ (since the identity map $I : X \to X$ is a homeomorphism)
if $X \cong Y$, then $Y \cong X$ (when $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$).
if $X \cong Y$ and $Y \cong Z$, then $X \cong Z$ (indeed, if $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is $h = g \circ f : X \to Z$ with $h^{-1} = (g \circ f)^{-1} = f^{-1} \circ g^{-1}$).

Example

$f : [0, 2p) \to S^1, f(t) = e^{it}$.

Theorem 1.13 Restriction of Domain and Restriction or Extension of Codomain

Let X, Y, Z be topological spaces. Suppose $f : X \to Y$ is continuous.

1. For any subspace $A \subseteq X$, then restriction $f : A \to Y$ is continuous.
2. If $Y \subseteq Z$ is a subspace, then $f : Y \to Z$ is continuous, and if $B \subseteq Y$ with $f(X) \subseteq B$, then $f : X \to B$ is continuous.

Proof: Exercise □

Lemma 1.14 The Gluing Lemma

Let $f : X \to Y$ be a map between topological spaces. Then

1. If $X = \cup_{k \in K} U_k$, where K is a non-empty set and each U_k is open in X, and if each restriction map $f|_{U_k} : U_k \to Y$ is continuous (where U_k is using the subspace topology), then f is continuous.
2. If $X = \cup_{k \in K} A_k$, where K is a non-empty finite set, and each A_k is closed in X, and if each restriction $f : A_k \to Y$ is continuous (where A_k is using the subspace topology), then f is continuous.

Proof: (1) Suppose $X = \cup_{k \in K} U_k$, where each U_k is open in X and suppose each restriction $g_k : U_k \to Y$ is continuous (where $g_k(x) = f(x)$ for all $x \in U_k$). Let $V \subseteq Y$ be open in Y. Note that

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \{x \in U_k \mid f(x) \in V\} \\ &= \bigcup_{k \in K} \{x \in U_k \mid g_k(x) \in V\} \\ &= \bigcup_{k \in K} g_k^{-1}(V) \end{aligned}$$

For each $k \in K$, since $g_k : U_k \to Y$ is continuous, we know that $g^{-1}(V)$ is open in U_k. Since U_k is using the subspace topology, we can choose an open set W_k in X such that $g_k^{-1}(V) = W_k \cap U_k$. This is open in X since W_k and U_k are both open in X. Since $f^{-1}(V) = \cup_{k \in K} g_k^{-1}(V)$, it is a union of open sets in X, so it is open in X. Thus, f is continuous (on X).

(2) First, show that for $f : X \to Y$, f is continuous (on X) $\Leftrightarrow$ $f^{-1}(C)$ is closed in X for every closed C in Y. Then show that when $A \subseteq X \subseteq Y$, A is closed in X (using the subspace topology from Y) $\Leftrightarrow$ $A = B \cap X$ for some closed set B in Y. □

Example

The map $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2x \text{ if } x \leq 0 \\ x^2 \text{ if } x \geq 0 \end{cases}$$

is continuous (by part (2) of the Gluing lemma)

Example

The circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is homeomorphic to the ellipse

$$\frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1$$

in $\mathbb{R}^2$. Indeed, the map $\varnothing : S \to E$ given by $\varnothing(x, y) = \left(\frac{x-a}{A}, \frac{y-b}{B}\right)$ and its inverse $\varphi = \varnothing^{-1} : E \to S$ given by $\varphi(u, v) = (Au + a, Bv + b)$

Example

$\mathbb{R} \cong (-1, 1) \subseteq \mathbb{R}$. The map $g : \mathbb{R} \to (-1, 1)$ given by $g(x) = \arctan(x)$ is a homeomorphism.

Example

The standard unit n-sphere in $\mathbb{R}^{n+1}$ is the set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, where p is the north pole $p = e_{n+1} = (0, ..., 0, 1) \in \mathbb{S}^n$. We have $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$. Indeed, we can use the stereographic projection map.

2 Examples of Topological Spaces

Definition 2.0.1

Let X be a set.
We sometimes write X_t to indicate that X is using the trivial topology $\mathcal{T}_t = \{\varnothing, X\}$.
We sometimes write X_d to indicate that X is using the discrete topoology

$$\mathcal{T}_d = P(X) = \{A \mid A \subseteq X\}$$

We sometimes write X_c to indicate that X is using the co-finite topology

$$\mathcal{T}_c = \{A \subseteq X \mid A = \varnothing \vee X \setminus A \text{ is finite}\}$$

Remark

The closed sets in $\mathcal{T}_c$ are the finite sets and the set X. $\mathcal{T}_c$ is the coarsest topology where the singletons are closed.

Definition 2.0.2

When X is a metric space, we assume, unless otherwise indicated, that X uses the metric topology. Sometimes, we might write X_m to indicate that X is using the metric topology $\mathcal{T}_m$.

Definition 2.0.3

When Y is a topological space and $X \subseteq Y$ is a subset, we assume, unless otherwise indicated, that X uses the subspace topology. Sometimes, we might write X_s to indicate that X is using the subspace topology $\mathcal{T}_s$.

Note

When $X \subseteq \mathbb{R}^n$, we shall assume, unless otherwise indicated, that X is using $\mathcal{T}_m = \mathcal{T}_s$.

Definition 2.0.4

Let X be a set. A (strict, linear or total) **order** on X is a binary relation $<$ on X which satisfies:

1. For all $x, y \in X$, exactly one of the following holds:
   a. $x < y$
   b. $x = y$
   c. $y < x$
2. For all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

Definition 2.0.5

An **ordered set** is a set X with an order $<$. When X is an ordered set, we also define $\leq, >, \geq$ by stipulating that for all $x, y \in Z$,

1. $x \leq y \iff x < y \vee x = y$
2. $x > y \iff y < x$
3. $x \geq y \iff y \leq x$

Remark

In an ordered set X, we can define an upper bound, a lower bound, the supremum, the infemum, the maximum and the minimum of a subset $A \subseteq X$.

Example

When X is an ordered set, and $A \subseteq X$ is a subset, $M = \max(A)$ when $M \in A$ with $M \geq x$ for all $x \in X$. $m = \min(A)$ when $m \in A$ with $m \leq x$ for all $x \in A$.

Definition 2.0.6

When X is an ordered set, we have the following subsets (which are called intervals in X): For $a, b \in X$ with $a < b$, we have
1. $(a, b) = \{x \in X \mid a < x < b\}$
2. $(a, b] = \{x \in X \mid a < x \leq b\}$
3. $[a, b) = \ldots$
4. $[a, b] = \ldots$

Example

In $\mathbb{Z}$ with its standard order, $(1, 5) = [2, 5) = (1, 4] = [2, 4] = \{2, 3, 4\}$.

Definition 2.0.7

Let X be an ordered set. The **order topology** on X is the topology $\mathcal{T}_o$, which is generated by the basis $\mathcal{B}_o$ of sets in X which consists of the following intervals:
1. (a, b) where $a, b \in X$ with $a < b$.
2. $(a, M]$ where $M = \max(X)$ and $a \in X$ with $a \neq M$ (in the case that X has a maximum M).
3. $[m, b)$ where $m = \min(X)$ and $b \in X$ with $b \neq m$ (in the case that X has a minimum).

Exercise 2.0.1

Verify that $\mathcal{B}_o$ is a basis of sets in X.

Proof: Clearly,

$$X = \bigcup_{a<b\in X} (a, b)$$

covers X if X has neither a max or a min. If X has a max and/or a min, include the intervals $(a, M]$ and $[m, b)$ in the union for any $a < M$ and/or $b > m$. $\square$

Notation

We sometimes write X_o to indicate that X is using the order topology.

Example

$\mathbb{R} = \mathbb{R}_o = \mathbb{R}_m$.

Definition 2.0.8

Let X be an ordered set. The **lower limit** topology on X is the topology T_l generated by the basis of sets $\mathcal{B}_l$, which consists of intervals of the form $[a, b)$, or of the form $[c, \max(X)]$, where $a, b \in X$, $c \in X \setminus \{\max(X)\}$ with $a < b$.

Note

We sometimes write X_l to indicate that X is using the lower limit topology.

Example

On $\mathbb{R}$, $T_0 \subseteq T_m$ because sets in T_0 are open intervals and open intervals are open in $\mathbb{R}$'s metric topology. Also, $T_m \subseteq T_0$ by the following argument: open sets in $\mathbb{R}$'s metric topology are unions of open intervals because the set of open intervals is a basis of sets on $\mathbb{R}$ that generates T_m. But open intervals are in T_0 and T_0 is closed under arbitrary unions. So $T_m \subseteq T_0$ and we have that $T_0 = T_m$.
However, T_l is not equal to T_m. Note that when $a, b \in \mathbb{R}$ with $a < b$,

$$(a, b) = \bigcup_{n=m}^{\infty} \left[a + \frac{1}{n}, b\right)$$

where $\frac{1}{m} < b - a$, which is open in $\mathcal{R}_l$, so we have $\mathcal{T}_0 \subseteq \mathcal{T}_l$.

Example

Let $X = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. Note that $\mathcal{T}_0 \neq \mathcal{T}_m = \mathcal{T}_s$ on X, where X uses the standard order inherited from $\mathbb{R}$. For example, $\{2\}$ is open in X_m, since $B_X(2, \frac{1}{2}) = B_{\mathbb{R}}(2, \frac{1}{2}) \cap X = \{2\}$. But $\{2\}$ is not open in X_o because any open set in X_o which contains 2, must contain a basic open set $\mathcal{B}$ with $2 \in \mathcal{B} \subseteq U$, so it must contain a set of the form $\mathcal{B} = (1, 2]_X = (a, 1) \cup \{2\}$, where $a \in (0, 1)$. So they must include elements other than 2.

Example

When X is an ordered set, the dictionary (or lexicographic) order on X^2 is given by stipulating that

$$(a, b) < (c, d) \Leftrightarrow \begin{cases} a = c \text{ and } b < d \\ a < c \end{cases}$$

Note

On $\mathbb{R}^2$, the order topology $\mathcal{T}_0$ is not equal to the standard metric topology $\mathcal{T}_m$.

2.1 Products of Topological Spaces

Definition 2.1.1

Let K be a non-empty set and let X_k be a set for each $k \in K$. The cartesian product of the (indexed set of) sets X_k, $k \in K$ is the set

$$\prod_{k \in K} X_k = \left\{ x : K \to \bigcup_{k \in K} X_k \mid x(k) \in X_k \text{ for all } k \in K \right\}$$

and we write $x(k)$ as x_k. In the case that $K = \{1, \ldots n\}$, we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{n} X_k = X_1 \times X_2 \times \ldots \times X_n$$

In the case that $K = \mathbb{Z}^+ = \{1, 2, \ldots\}$, we write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} X_k = X_1 \times X_2 \times X_3 \times \ldots$$

In the case that $K = \{1, 2, \ldots, n\}$ and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k \in K} X = X \times X \times \ldots \times X = X^n$$

In the case that $K = \mathbb{Z}^+ = \{1, 2, 3 \ldots\}$ and $X_k = X$ for all $k \in K$, we also write

$$\prod_{k \in K} X_k = \prod_{k=1}^{\infty} X \times X \times X \times \ldots = X^\omega$$

Remark

We remark that the n-tuple $x = (x_1, x_2, ..., x_n) \in X_1 \times ... \times X_n$ is (by definition) the function

$$x : \{1, 2, ...n\} \to \bigcup_{k=1}^{n} X_k$$

such that $x_k = x(k) \in X_k$ and that the sequence

$$x = (x_n)_{n \geq 1} = (x_1, x_2, x_3, ...) \in X_1 \times X_2 \times X_3 \times ...$$

is (by definition) equal to the function

$$x : \mathbb{Z}^+ \to \bigcup_{k=1}^{\infty} X_k$$

given by $x(k) = x_k$.

Notation

In the special case that X is a vector space, we also write

$$X^\infty = \{x \in X^\omega \mid x_k = 0 \text{ for all but finitely many } k \in \mathbb{Z}^+\}$$

In this case, both X^∞ and X^ω are vector spaces. $\dim(\mathbb{R}^\infty) = \aleph_0$ and $\mathbb{R}^\omega = 2^{\aleph_0}$. $\mathbb{R}^\infty$ has standard basis $\{e_1, e_2, e_3, ...\}$.

Example

When X_k is a set for each $k \in K$, for each $l \in K$, we have the projection map

$$p_l : \prod_{k \in K} X_k \to X_l$$

given by $p_l(x) = x_l := x(l)$. For any set Y, a function

$$f : Y \to \prod_{k \in K} X_k$$

determines, and is determined by, its component functions.
$f_l : Y \to X_l$, where $f_l = p_l \circ f$, so $f_l(y) = p_l(f(y)) = f(y)_l = f(y)(l)$.

Definition 2.1.2

When X_k is a topological space for each $k \in K$, there are two commonly used topologies on $\prod_{k\in K} X_k$:

The *box topology* on $\prod_{k\in K} X_k$ is the topology generated by the basis of sets of the form

$$\prod_{k\in K} U_k \subseteq \prod_{k\in K} X_k$$

where each U_k is open in X_k.

The *product topology* on $\prod_{k\in K} X_k$ is the topology generated by the basis of sets consisting of the sets of the form $\prod_{k\in K} U_k$, where each U_k is open in X_k with $U_k = X_k$ for all but finitely many $k \in K$.

Note

The above two proposed bases are indeed bases of sets because

$$\left(\prod_{k\in K} U_k\right) \cap \left(\prod_{k\in K} V_k\right) = \prod_{k\in K} (U_k \cap V_k)$$

Also note that when K is finite, these two topologies are equal. When K is infinite, the box topology is finer than the product topology.

Theorem 2.1

Let $\mathcal{B}_k$ be a basis for X_k for each $k \in K$. Then the set of sets of the form

$$\prod_{k\in K} B_k$$

where $B_k \in \mathcal{B}_k$ for all $k \in K$ is a basis for the box topology on $\prod_{k\in K} X_k$, where the set of sets of the form

$$\prod_{k\in K} B_k$$

, where $B_k \in \mathcal{B}_k \cup \{X_k\}$ for all $k \in K$ with $B_k = X_k$ for all but finitely many $k \in K$, is a basis for the product topology on $\prod_{k\in K} X_k$.

Proof:

1. First, we prove that the set in question is a basis for the box topology on $\prod_{k\in K} X_k$. Let U in the box topology be arbitrary. Then for all $k \in K$, $p_k(U)$ is open in X_k.

□

Theorem 2.2

For each $k \in K$, let X_k be a subspace of Y_k (using the subspace topology). Then the box topology on $\prod_{k\in K} X_k$ is equal to the subspace toplogy on $\prod_{k\in K} X_k$ as a subspace of $\prod_{k\in K} Y_k$ using the box topology, and the product topology on $\prod_{k\in K} X_k$ is equal to the subspace topology on $\prod_{k\in K} X_k$ as a subspace of $\prod_{k\in K} Y_k$ using the product topology.

Proof: Any set in the box topology on $\prod_{k\in K} X_k$ is of the form $\prod_{k\in K} U_k$, where U_k is open in X_k. Since X_k inherits the subspace topology from Y_k, then $U_k = V_k \cap X_k$ for some V_k, which is open in Y_k. So

$$\prod_{k\in K} U_k = \prod_{k\in K} (V_k \cap X_k) = \prod_{k\in K} V_k \cap \prod_{k\in K} X_k$$

where $\prod_{k\in K} V_k$ is clearly open in the box topology on $\prod_{k\in K} Y_k$. So $\prod_{k\in K} U_k$ is open in the subspace topology on $\prod_{k\in K} X_k$ inherited from the topology on $\prod_{k\in K} Y_k$. The other direction of the proof applies the same logic, so it is omitted. Now, for the product topology, any set in the product topology on $\prod_{k\in K} X_k$ is of the form $\prod_{k\in K} U_k$, where for all but finitely many $k \in K$, we have $U_k = X_k$. Since X_k inherits the subspace topology from Y_k, then $U_k = V_k \cap X_k$, where V_k is some open set in Y_k. So

$$\prod_{k\in K} U_k = \prod_{k\in K} (V_k \cap X_k) = \prod_{k\in K} V_k \cap \prod_{k\in K} X_k$$

where contd. □

Theorem 2.3

Let Y be a topological space and let X_k be a topological space for each $k \in K$, and let $f : Y \to \prod_{k\in K} X_k$. Then when $\prod_{k\in K} X_k$ uses the product topology, f is continuous $\iff$ each component map $f_l : Y \to X_l$ (given by $f_l(y) = f(y)_l = f(y)(l)$) is continuous.

Proof: ($\Longrightarrow$) Suppose that f is continuous. Then (using either the box or product topologies on $\prod_{k\in K} X_k$) each projection map $p_l : \prod_{k\in K} X_k \to X_l$ is continuous because when $U \subseteq X_l$ is open, $p_l^{-1}(U) = \left\{x \in \prod_{k\in K} X_k \mid x_l = p_l(x) \in U\right\} = \prod_{k\in K} U_k$, where

$$U_k = \begin{cases} U \text{ if } k = l \\ X_k \text{ if } k \neq l \end{cases}$$

which is open in $\prod_{k\in K} X_k$ (using either the box or product topology). It follows that each component function f_l is continuous because $f_l = p_l \circ f$.
($\Longleftarrow$) Suppose that each component map $f_l := p_l \circ f : Y \to X_l$ is continuous and that $\prod_{k\in K} X_k$ uses the product topology. To show that f is continuous, it suffices to show that $f^{-1}(B)$ is open in Y for every basic open set B in $\prod_{k\in K} X_k$.
Let B be a basic open set for the product topology on $\prod_{k\in K} X_k$. Say $B = \prod_{k\in K} U_k$,

where each U_k is open in X_k with $U_k = X_k$ for all but finitely many indices $k \in K$. Let $L \subseteq K$ be the finite set of all indices $k \in K$ for which $U_k \neq X_k$. We have

$$\begin{aligned} f^{-1}(B) &= \left\{ y \in Y \mid f(y) \in \prod_{k \in K} U_k \right\} \\ &= \{ y \in Y \mid f_k(y) = f(y)_k \in U_k \text{ for all } k \in K \} \\ &= \{ y \in Y \mid f_k(y) \in U_k \text{ for all } k \in L \} \\ &= \bigcap_{k \in L} f_k^{-1}(U_k) \end{aligned}$$

which is open in Y since it is a finite intersection of open sets in Y (with $f_k^{-1}(U_k)$ is open in Y because U_k is open in X_k and $f_k : Y \to X_k$ is continuous). □

Recall

$$\mathbb{R}^\infty \subseteq l_1 \subseteq l_p \subseteq l_q \subseteq l_\infty \subseteq \mathbb{R}^\omega$$

for $1 \leq p \leq q \leq \infty$. $\mathbb{R}^\infty$ denotes the set of eventually zero sequences.
Recall or verify that the p-norms on $\mathbb{R}^\infty$ (or on l_1) induce different topologies on $\mathbb{R}^\infty$ or (l_1).
Question: do any of the p-norms induce the box or product topology on $\mathbb{R}^\infty \subseteq \mathbb{R}^\omega$?
Question: Is there a norm or metric on $\mathbb{R}^\omega$ which induces the box or product topology?
Also, we have the p-norms on $\mathbb{R}^n$. They all give the same topology on $\mathbb{R}^n$. More generally, when X is a finite dimensional vector space, all norms on X induce the same topology on X: when $L : X \to Y$ is a linear map between normed linear spaces, L is continuous $\Leftrightarrow \|L\|_{\text{op}} < \infty \Leftrightarrow L\left(\overline{B_X}(0,1)\right)$ is bounded in Y. When X is finite dimensional, $\overline{B_X}(0,1)$ is compact and $L\left(\overline{B_X}(0,1)\right)$ is bounded, then L is continuous. In particular, when X is finite dimensional and $\| \ \|_1$ and $\| \ \|_2$ are two norms on X, then $I : (X, \| \ \|_1) \to (X, \| \ \|_2)$ is continuous and it is equal to its own inverse $I^{-1}(X, \| \ \|_2) \to (X, \| \ \|_1)$, which is continuous. So I is a homomorphism. So for $U \subseteq X$, U is open in $(X, \| \ \|_1)$ if and only if U is open in $(X, \| \ \|_2)$.

Remark

Consequently, every finite dimensional vector space X has a standard topology. Pick any basis $\{u_1, ...u_n\}$, define

$$\left\langle \sum_{k=1}^{n} x_k u_k, \sum_{k=1}^{n} y_k u_k \right\rangle = \sum_{k=1}^{n} x_k y_k = x \cdot y$$

So the map $L : X \to \mathbb{R}^n$ given by

$$L\left(\sum_{k=1}^{n} x_k u_k\right) = \sum_{k=1}^{n} x_k e_k = x$$

is an inner product space isomorphism. Then use the inner product to define a norm, a metric and a topology. The resulting topology does not depend on the choice of basis.

2.2 Quotients Spaces

Definition 2.2.1

Let X be a set and let $\sim$ be an equivalence relation on X. For $a \in X$, the *equivalence class* of a (under $\sim$) is the set

$$[a] = \{x \in X \mid a \sim x\}$$

Recall that distinct equivalence classes are disjoint and X is the (disjoint) union of the (distinct) equivalence classes. The set of all equivalence classes is denoted by

$$X/\sim$$

and is called the quotient (set) of X by $\sim$.

$$X/\sim = \{[a] \mid a \in X\}$$

The map $q : X \to X/\sim$ given by $q(x) = [x]$ is called the *quotient map* (for $X/\sim$).

Definition 2.2.2

When X is a topological space, the *quotient toplogy* on $X/\sim$ is the topology obtained by stipulating that for $V \subseteq X/\sim$, V is open in $X/\sim \iff q^{-1}(V)$ is open in X. This gives a topology because

1. $q^{-1}(\varnothing) = \varnothing$
2. $q^{-1}(X/\sim) = X$, which is open in X
3. for an arbitrary collection $\{V_k\}_{k\in K}$ open sets in $X/\sim$,
$$q^{-1}\left(\bigcup_{k\in K} V_k\right) = \bigcup_{k\in K} q^{-1}(V_k)$$
4. for any finite collection $\{V_k\}_{k\in K}$ of open sets in $X/\sim$,
$$q^{-1}\left(\bigcap_{k\in K} V_k\right) = \bigcap_{k\in K} q^{-1}(V_k)$$

Note

When $V \subseteq X/\sim$, so V is a set of equivalence classes, then

$$\begin{aligned} q^{-1}(V) &= \{x \in X \mid q(x) \in V\} \\ &= \{x \in X \mid [x] \in V\} \\ &= \bigcup_{[x]\in V} [x] = \cup V \end{aligned}$$

Remark

For topological spaces X and Y,

1. When $X \subseteq Y$ is a subset, the subset topology is the *coarsest* topology on X for which the inclusion map $\int : X \to Y$ (given by $\int(x) = x$) is continuous. To see, this observe that for V open in Y,
$$\int^{-1}(V) = \left\{x \in X \mid \int(x) \in V\right\} = \{x \in X \mid x \in V\} = V \cap X$$
.
2. The product topology on $X \times Y$ is the *coarsest* topology on $X \times Y$ for which the two projection maps $f_X : X \times Y \to X$, $f_Y : X \times Y \to Y$ are both continuous ($f_X^{-1}(U) = U \times Y$, $f_Y^{-1}(V) = X \times V$).
3. When $\sim$ is an equivalence relation on X, the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ for which the quotient map $q : X \to X/\sim$ is continuous.

Note

Let X be a set and let $\sim$ be an equivalence relation. Note that any function $g : X/\!\sim\, \to Y$ (where Y is any set) determines and is determined by a function $f : X \to Y$, which is constant on equivalence classes (meaning that for $x_1, x_2 \in X$, if $x_1 \sim x_2$, then $f(x_1) = f(x_2)$):
with g given by $g([x]) = f(x)$ and with f given by $f = g \circ q$, so $f(x) = g(q(x)) = g([x])$.

Theorem 2.4

Let X be a topological space. Let $\sim$ be an equivalence relation on X and let $f : X/\!\sim\, \to Y$. Let $g : X \to Y$ be the map given by $g(x) = f([x])$, that is, $g = f \circ q$. Then f is continuous $\iff g$ is continuous.

Proof: If f is continuous, then g is continuous because $g = f \circ q$, which is the composite of two continuous maps. Suppose that g is continuous. Let $V \subseteq Y$ be open. We need to show that $f^{-1}(V)$ is open in $X/\!\sim$. By definition of the quotient topology, $f^{-1}(V)$ is open in $X/\!\sim\, \iff q^{-1}(f^{-1}(V))$ is open in X. But $q^{-1}(f^{-1}(V)) = (f \circ q)^{-1}(V) = g^{-1}(V)$, which is open in X since g is continuous. $\square$

Definition 2.2.3

For a group G and set X, a *group action* of G on X is a function $\star : G \times X \to X$, which we write $\star\,(a, x)$ as $a \star x$, or simply as ax, such that

1. When $e \in G$ is the identity element, we have $e \star x = x$ for all $x \in X$.
2. For all $a, b \in G$ and all $x \in X$, we have $a \star (b \star x) = (ab) \star x$.

We say that G acts on X by the group action.

Remark

A group action of G on X determines and is determined by a group homomorphism $\rho : G \to \operatorname{Perm}(X)$, where $\rho(a)(x) = a \star x$ (the homomorphism ρ is called a *representation* of G).
Given an action of G on X, we can define an equivalence relation on X by

$$x \sim y \Leftrightarrow y = a \star x \text{ for some } a \in G$$

In this case, the equivalence class of x is called the orbit of x (we might write $[x]$ as Orb(x)) and we write the quotient $X/\!\sim$ as X/G. So

$$\begin{aligned} X/G &= \{[x] \mid x \in X\} \\ &= \{\operatorname{Orb}(x) \mid x \in X\} \end{aligned}$$

Example

For $\mathbb{S}^1 = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$, we have $\mathbb{S}^1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$.
Proof: Define $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ by

$$f(u, t) = e^t u$$

and define $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1 \times \mathbb{R}$ by

$$g(x) = \left(\frac{x}{\|x\|}, \ln(\|x\|) \right)$$

These maps are continuous (they are elementary functions) and they are inverses of each other. Indeed, $f(g(x)) = g(f(x)) = x$. $\square$

Example

$\mathbb{S}^1$ acts on $\mathbb{R}^2 \setminus \{0\} = \mathbb{C}^*$ by (complex) multiplication. For $a \in \mathbb{R}^2 = \mathbb{C}$,

$$\mathrm{Orb}(a) = [a] = \left\{ua \mid u \in \mathbb{S}^1\right\}$$

which is equal to the circle centered at 0 of radius $\|a\|$ (with $[0] = \{0\}$). Show that

$$\mathbb{R}^2/\mathbb{S}^1 \cong [0, \infty) \subseteq \mathbb{R}$$

We define

$$f : \mathbb{R}^2/\mathbb{S}^1 \to [0, \infty)$$

by $f([x]) = \|x\|$ and define

$$h : [0, \infty) \to \mathbb{R}^2/\mathbb{S}^1$$

by $h(r) = [r] = \left\{re^{i\theta} \mid \theta \in \mathbb{R}\right\}$.

Note that $f : \mathbb{R}^2/\mathbb{S}^1 \to [0, \infty) \subseteq \mathbb{R}$ is continuous because for the map $g : \mathbb{R}^2 \to [0, \infty) \subseteq R$ given by $g(x) = \|x\|$, we have $g = f \circ q$. Since g is continuous (it's elementary), it follows that f is continuous. Also, it follows that h is continuous because $h = q \circ i$, where $i : [0, \infty) \to \mathbb{R}^2$ is the inclusion map $i(r) = (r, 0)$. Finally, we note that f and h are inverses by direct verification.

Example

$\mathbb{R}^+ = (0, \infty)$ acts on $\mathbb{R}^2$ by multiplication (or by scaling, that is by $t \star x = tx$). The orbits are for $0 \neq x \in \mathbb{R}^2$,

$$[x] = \{tx \mid 0 < t \in \mathbb{R}\}$$

which is the (open) ray from 0 through x and

$$[0] = \{t \cdot 0 \mid 0 < t \in \mathbb{R}\} = \{0\}$$

Each of the rays $[x]$ for $0 \neq x \in \mathbb{R}^2$ intersects a unique point in $\mathbb{S}^1$, and this gives a fairly natural bijective map

$$f : \mathbb{R}^2/\mathbb{R}^+ \to \mathbb{S}^1 \cup \{0\}$$

given by

$$f([x]) = \begin{cases} \frac{x}{\|x\|} \text{ if } 0 \neq x \in \mathbb{R}^2 \\ 0 \text{ if } x = 0 \in \mathbb{R}^2 \end{cases}$$

The inverse $g : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2/\mathbb{S}^1$ is given by $g(u) = [u]$.
Note that g is continuous ($g = q \circ k$, where k is the inclusion map

$$k : \mathbb{S}^1 \cup \{0\} \to \mathbb{R}^2$$

).
But f is not continuous. For example, the set $V = \{0\}$ is open in $\mathbb{S}^1 \cup \{0\}$ (indeed, $\{0\} = B(0, 1)$) but $f^{-1}(V) = [0] = \{0\} \subseteq \mathbb{R}^2/\mathbb{R}^+$ and $q^{-1}(\{0\}) = \{0\}$, which is not open in $\mathbb{R}^2$. In fact, $\mathbb{R}^2/\mathbb{R}^+ \ncong S^1 \cup \{0\}$. One way to show this is to note that in $\mathbb{S}^1 \cup \{0\}$, the 1-point set $\{0\}$ is open, but $\{[0]\}$ is not open in $\mathbb{R}^2/\mathbb{R}^+$.

Remark

$\mathbb{R}^2/\mathbb{R}^+$ is not Hausdorff, so it is not metrizable (there is no metric we can define on $\mathbb{R}^2/\mathbb{R}^+$ for which the quotient topology is equal to the metric topology).

Example

$\mathbb{Z}$ acts by addition on $\mathbb{R}$ (by $n \star x = x + n$). The orbits are the sets $[x] = \{x + n \mid n \in \mathbb{Z}\} = x + \mathbb{Z}$. Show that $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$.
Define $f : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ by $f([t]) = e^{i2\pi t} = (\cos 2\pi t, \sin 2\pi t)$ (and note that when $[s] = [t]$, say $s = t + n$, where $n \in \mathbb{Z}$, we have $e^{i2\pi s} = e^{i2\pi(t+n)} = e^{i2\pi t}$). Note that f is continuous because the map $g : \mathbb{R} \to \mathbb{S}^1$ given by $g(t) = e^{i2\pi t}$ is continuous (it's elementary) with $g(t) = f([t])$, that is, $g = f \circ q$. The inverse map $h : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z}$ is given by $h(e^{i\theta}) = \left[\frac{\theta}{2}\pi\right]$. To see that h is continuous, we can express h in Cartesian coordinates.

Remark

There is an angle map

$$\Theta : \mathbb{R}^2 \setminus \{0\} \to [0, 2\pi)$$

given by

$$\Theta(x, y) = \begin{cases} \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \text{ if } y \geq 0 \text{ or } (y = 0 \land x \neq 0) \\ 2\pi - \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \text{ if } y < 0 \text{ or } (y = 0 \land x < 0) \end{cases}$$

This map Θ is not continuous along the positive x-axis.

Example

In cartesian coordinates, $h : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z}$ is given by

$$h(x, y) = \begin{cases} \left[\frac{1}{2\pi}\cos^{-1}(x)\right] \text{ if } y \geq 0 \\ \left[1 - \frac{1}{2\pi}\cos^{-1}(x)\right] \text{ if } y \leq 0 \end{cases}$$

That is, by

$$h(x, y) = \begin{cases} h_1(x, y) \text{ if } (x, y) \in A \\ h_2(x, y) \text{ if } (x, y) \in B \end{cases}$$

where $A = \{(x, y) \in \mathbb{S}^1 \mid y \geq 0\}$, $B = \{(x, y) \in \mathbb{S} \mid y \leq 0\}$ and $h_1(x, y) = \frac{1}{2\pi}\cos^{-1}(x)$ and $h_2(x, y) = 1 - \frac{1}{2\pi}\cos^{-1}(x)$.

3 Connectedness, Path Connectedness and Compactness

3.1 Connectedness and Connected Components

Definition 3.1.1

Let X be a topological space. For subsets $A, B \subseteq X$, we say that A and B *separate* X when $A \neq \varnothing$, $B \neq \varnothing$, $A \cap B = \varnothing$ and $A \cup B = X$.

Definition 3.1.2

We say that X is *disconnected* when there exist (non-empty disjoint) open sets $U, V \subseteq X$, which separate X. Otherwise, we say that X is connected.

Remark

Note that X is connected if and only if the only subsets of X which are both open and closed are the sets $\varnothing$ and X. Indeed, if X is disconnected and $U, V \subseteq X$ are open sets in X which separate X (so $U \neq \varnothing, V \neq \varnothing, U \cap V = \varnothing, U \cup V = X$). Then the sets $\varnothing, U, V, X$ are both open and closed since $U = V^c = X \setminus V$ and $V = U^c = X \setminus U$. On the other hand, if $\varnothing \neq U \subseteq X$ with U both open and closed in X, then U and $V = U^c = X \setminus U$ are open sets in X which separate X.

Exercise 3.1.1

Recall, or verify, that when X is a metric space and $A \subseteq X$ is a subspace, A is connected if and only if there do not exist open sets U, V in X such that $U \cap A \neq \varnothing, V \cap A \neq \varnothing, U \cap V = \varnothing$ and $A \subseteq U \cup V$.

Exercise 3.1.2

Recall, or verify, that the connected sets in $\mathbb{R}$ are the intervals (including $\varnothing$, $\{x\}, \mathbb{R}$).

Exercise 3.1.3

Recall, or verify, that the intervals in $\mathbb{R}$ are the sets $A \subseteq \mathbb{R}$ with the intermediate value property ($\forall a, b, c \in \mathbb{R}$, if $a < b < c$ and $a, c, \in A$, then $b \in A$).

Example

The non-empty connected subsets of $\mathbb{Q}$ are the 1-point sets.
1-point sets are connected vacuously and if $A \subseteq \mathbb{Q}$ has at least 2 points, say $a, b \in \mathbb{Q}$ with $a < b$, choose $r \notin \mathbb{Q}$ with $a < r < b$. Then $U = \{x \in \mathbb{Q} \mid x < r\}$ and $V = \{x \in \mathbb{Q} \mid x > r\}$ are open sets in $\mathbb{Q}$ which separate $\mathbb{Q}$.

Theorem 3.1

If $f : X \to Y$ is a *continuous* map of topological spaces, and if X is connected, then $f(X)$ is connected (as a subspace of Y).

Proof: Suppose X is connected and $f : X \to Y$ is continuous. By restricting the codomain, the map $f : X \to f(X)$ is also continuous. Suppose, for a contradiction, that $f(X)$ is disconnected. Let U and V be open sets in $f(X)$ which separate $f(X)$ (so $U \neq \varnothing, V \neq \varnothing, U \cap V = \varnothing, U \cup V = X$). Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in X which separate X, so that X is disconnected, giving the desired contradiction ($f^{-1}(U)$ and $f^{-1}(V)$ are open because f is continuous and $f^{-1}(U) \neq \varnothing$ since $U \neq \varnothing$ and $f^{-1}(V) \neq \varnothing$ since $V \neq \varnothing$, and $f^{-1}(U) \cap f^{-1}(V) \neq \varnothing$). □

Lemma 3.2

Let X be a subspace of Y. Suppose Y is disconnected. Let U and V be open sets in Y which separate Y. If X is connected, then $X \subseteq U$ or $X \subseteq V$.

Proof: Suppose that $X \nsubseteq U$ and $X \nsubseteq V$. Since $U \cup V = Y$, it follows that $X \cap U \neq \varnothing$ and $X \cap V \neq \varnothing$ (since $X \nsubseteq V$, so $X \cap V^c \neq \varnothing$, where $V^c = Y \setminus V = U$) and also, these two sets are open sets in X which separate X
$(X \cap U \neq \varnothing, X \cap V \neq \varnothing, (X \cap U) \cap (X \cap V) = X \cap (U \cap V) = X \cap \varnothing = \varnothing,$
$(X \cap U) \cup (X \cap V) = X \cap (U \cup V) = X \cap Y = X)$. □

Theorem 3.3

Let $X = \cup_{k \in K} A_k$, where each subspace A_k is connected with $\cap_{k \in K} A_k \neq \varnothing$. Then X is connected.

Proof: Suppose, for a contradiction, that X is disconnected. Let U and V be open sets in X which separate X. For each $k \in K$, since A_k is connected, either $A_k \subseteq U$ or $A_k \subseteq V$ (by the previous lemma). Let $p \in \cap_{k \in K} A_k \subseteq U \cup V$. Either $p \in U$ or $p \in V$ (but not both since $U \cap V = \varnothing$). Say $p \in U$ (so $p \notin V$ since $U \cap V = \varnothing$). So we must have $A_k \subseteq U$. Since $A_k \subseteq U$ for every $k \in K$, we have $X = \cup_{k \in K} A_k \subseteq U$. This is not possible since U and V separate X (so $V = U^c = X \setminus U \neq \varnothing$). □

Theorem 3.4

The (cartesian) product of two connected (topological) spaces is connected.

Proof: Let X and Y be two connected topological spaces. If $X = \varnothing$ or $Y = \varnothing$, then $X \times Y = \varnothing$, which is connected. Suppose $X \neq \varnothing$ and $Y \neq \varnothing$. Choose $a \in X$ and $b \in Y$ so $(a, b) \in X \times Y$. Since $X \times \{b\} \cong X$ and X is connected, it follows that $X \times \{b\}$ is connected (since the image of a connected space under a continuous map is connected). For each $x \in X$, since $\{x\} \times Y \cong Y$ and Y is connected, then $\{x\} \times Y$ is connected. Since $X \times \{b\}$ and $\{x\} \times Y$ are connected and $(X \times \{b\}) \cap (\{x\} \times Y) \neq \varnothing$ (since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$), it follows from the previous theorem that the set $A_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected. Since each A_x is connected and $\cap_{x \in X} A_x \neq \varnothing$ (indeed $(a, b) \in A_x$ for all $x \in X$), it follows that $\cup_{x \in X} A_x$ is connected. Finally, note that $\cup_{x \in X} A_x = X \times Y$, so that $X \times Y$ is connected. □

Lemma 3.5

Let X be a subspace of Y. Let U and V be subsets of X (not necessarily open) which separate X ($U \neq \varnothing, V \neq \varnothing, U \cap V = \varnothing, U \cup V = X$). Then U is open in $X \iff U \cap \overline{V} = \varnothing$ and V is open in $X \iff V \cap \overline{U} = \varnothing$, where $\overline{U} = Cl_Y(U), \overline{V} = Cl_Y(V)$.

Proof: U is open in X. So V is closed in X (since $V = U^c = X \setminus U$). So

$$\begin{aligned} V = Cl_X(V) &= \cap \{K \mid K \subseteq X \text{ closed in } X \text{ with } V \subseteq K\} \\ &= \cap \{L \cap X \mid L \subseteq Y \text{ is closed in } Y \text{ with } V \subseteq L\} \\ &= (\cap \{L \mid L \subseteq Y \text{ closed} \in Y \text{ with } V \subseteq L\}) \cap X \\ &= U_Y(V) \cap X \\ &= \overline{V} \cap X \end{aligned}$$

if and only if

$$\begin{aligned} V &= \overline{V} \cap X \\ &= \overline{V} \cap (U \cup V) \\ &= \left(\overline{V} \cap U\right) \cup \left(\overline{V} \cap V\right) \\ &= \left(\overline{V} \cap U\right) \cup V \\ &= \varnothing \end{aligned}$$

since $U \cap V = \varnothing$, so that $\overline{V} \cap U$ is disjoint from V. □

Theorem 3.6

Let X be a topological space, let $A, B \subseteq X$ be subspaces with $A \subseteq B \subseteq \overline{A}$. If A is connected, then so is B. In particular, if A is connected, then so is $\overline{A}$.

Proof: Suppose that A is connected. Suppose, for a contradiction, that B is disconnected. Let $U, V \subseteq B$ be open sets in B which separate B
($U \neq \varnothing, V \neq \varnothing, U \cap V = \varnothing, U \cup V = B$). Since A is connected, and U and V are open sets in B, which separate B, by a previous lemma, either $A \subseteq U$ or $A \subseteq V$. Say $A \subseteq U$. Since $A \subseteq U$, we have $\overline{A} \subseteq \overline{U}$ so that $B \subseteq \overline{A} \subseteq \overline{U}$. By the previous lemma, $V \cap \overline{U} = \varnothing$, so $V \cap B = \varnothing$. Let $V \subseteq B$ so $V = \varnothing$, which contradicts the fact that U and V separate B. □

Theorem 3.7

Let X_k be connected topological spaces for each $k \in K$. Then $\prod_{k \in K} X_k$ is connected using the product topology.

Proof: If $X_k = \varnothing$ for some $k \in K$, then $\prod_{k \in K} X_k = \varnothing$ (which is connected). Suppose that $X_k \neq \varnothing$ for all $k \in K$. For each $k \in K$, choose $a_k \in X_k$. Let $a \in \prod_{k \in K} X_k$ (so $a : K \to \cup_{k \in K} X_k$) be given by $a(k) = a_k$ for all $k \in K$. Let $\mathcal{S}$ be the set of all finite subsets of K. For each $J \in \mathcal{S}$, let

$$Y_J = \left\{ y \in \prod_{k \in K} X_k \mid y_k = a_k \text{ for all } k \notin J \right\} \subseteq \prod_{k \in K} X_k$$

We claim that $Y_J \cong \prod_{j \in J} X_j$ (where Y_J is using the subspace topology, and $\prod_{k \in K} X_k$ and $\prod_{j \in J} X_j$ are using the product topology). This is a fairly trivial map

$$f : Y_J \to \prod_{j \in J} X_j$$

given by $f(y)(j) = y_j$ (where $y \in Y_J$ and $j \in J$) with inverse

$$g = f^{-1} : \prod_{j \in J} X_j \to Y_J$$

given by

$$g(x)(k) = \begin{cases} x_k \text{ when } k \in J \\ a_k \text{ when } k \notin J \end{cases}$$

Note that f is continuous. Indeed, given open sets U_j in X_j for each $j \in J$, so that $\prod_{j \in J} U_j$ is a basic open set in $\prod_{j \in J} X_j$, we have

$$\begin{aligned} f^{-1}\left(\prod_{j \in J} U_j\right) &= \left\{ y \in Y_J \mid f(y) \in \prod_{j \in J} U_j \right\} \\ &= \left\{ y \in Y_J \mid y_j \in U_j \text{ for all } j \in J \right\} \\ &= \left\{ y \in Y_J \mid y \in \prod_{k \in K} Y_k \right\} \text{ where } U_k = X_k \text{ for } k \in K \setminus J \\ &= \prod_{k \in K} U_k \cap Y_J \end{aligned}$$

which is a basic open set in Y_J (using the subspace topology). Similarly, the inverse image under g of a basic open set is a basic open set, so that g is continuous. Thus $Y_J \cong \prod_{j \in J} X_j$ as claimed. Since each X_j is connected and J is finite, it follows that $\prod_{j \in J} X_j$ is connected (by the previous theorem and by induction, and for $J = \{j_1, \ldots, j_n\}$, $\prod_{j \in J} X_j \cong \left(X_{j_1} \times \ldots \times X_{j_n}\right)$). Since $Y_J \cong \prod_{j \in J} X_j$, it follows that Y_J is connected. Since Y_J is connected, and for every $J \in \mathcal{S}$, and $\cap_{J \in \mathcal{S}} Y_J \neq \varnothing$ (indeed, $a \in Y_h$ for all $J \in \mathcal{S}$). It follows that $\cup_{J \in \mathcal{S}} Y_J$ is connected. To finish the proof, show that

$$\overline{\bigcup_{J \in \mathcal{S}} Y_J} = \prod_{k \in K} X_k$$

□

Definition 3.1.3

When X is a topological space and $S \subseteq X$ is a subset, we say that A is *dense* in X when $\overline{A} = X$. Note that $\overline{A} = X \iff$ the only closed set $K \in X$ with $A \subseteq K$ $(A \cap K^c \neq \varnothing)$ is $K = X \iff$ the only open set U in X with $A \cap U = \varnothing$ is $U = \varnothing \iff$ for every non-empty open set U in X, we have $A \cap U \neq \varnothing$.
When $\mathcal{B}$ is a basis for the topology on X, verify that $\overline{A} = X \iff$ for all $\varnothing \neq B \in \mathcal{B}$, we have $A \cap B \neq \varnothing$.

To complete the proof of the theorem, it suffices to show that for every non-empty basic open set B in $\prod_{k \in K} X_k$, we have $\left(\cup_{J \in \mathcal{S}} Y_J\right) \cap B \neq \varnothing$. Let $I \subseteq \mathcal{S}$. For each $k \in K$, let U_k be open in X_k be open in X_k with $U_k = X_k$ for all $k \notin I$ (so that $\prod_{k \in K} U_k$ is a non-empty basic open set in $\prod_{k \in K} X_k$). Then $Y_I \cap \left(\prod_{k \in K} U_k\right) \neq \varnothing$ (since we can choose $y \in Y_I$ with $y_k \in U_k$ for all $k \in I$ and with $y_k = a_k$ for all $k \notin I$). Since $Y_I \subseteq \cup_{J \in \mathcal{S}} Y_J$, we also have $\cup_{J \in \mathcal{S}} Y_J \cap \prod_{k \in K} U_k \neq \varnothing$.

Example

$\mathbb{R}^\omega = \prod_{k=1}^{\infty} \mathbb{R}$ using the box topology is *not* connected. Indeed, we can verify that the sets $U = \{x \in \mathbb{R}^\omega \mid \|x\|_\infty < \infty\}$ = the set of all bounded sequences and $V = \{x \in \mathbb{R}^\omega \mid \|x\|_\infty = \infty\}$ = the set of unbounded sequences in $\mathbb{R}$, are open in $\mathbb{R}^\omega$ (with the box topology) and they cover $\mathbb{R}^\omega$.

3.2 Connected Components

Definition 3.2.1

Let X be a topological space. Define a binary relation $\sim$ on X by stipulating that for $a, b \in X$,

$$a \sim b \iff \text{there exists a connected subspace of } X \text{ with } a, b \in A$$

Note that $\sim$ is an equivalence relation. $a \sim a$ since $\{a\}$ is connected (obviously) and if $a \sim b$, then $b \sim a$ (obviously, from the definition of $\sim$), and if $a \sim b$ and $b \sim c$, then we can choose connected subspaces $A, B \subseteq X$ with $a, b \in A$ and $b, c \in B$ and then, by a previous theorem, since $b \in A \cap B$, it follows that $A \cup B$ is connected, and $a, c \in A \cup B$ so that $a \sim c$.
The equivalence classes in X under the above equivlance relation $\sim$ are called the *connected components* of X. (Note that the connected components are disjoint and they cover X)

Theorem 3.8

Let X be a topological space. The connected components of X are the maximal connected subspaces of X. Indeed, each connected component is connected, and every non-empty connected subspace of X is contained inside exactly one of the connected components of X.

Proof: We claim that every non-empty connected subspace of X is contained in exactly one of the connected components of X. Let $\varnothing \neq A \subseteq X$ be a connected subspace of X. Let $a \in A$ and let $C = [a]$ (and note that $A \cap C \neq \varnothing$). Let D be any connected component of X with $A \cap D \neq \varnothing$. Let $b \in A \cap D$. Then since A is connected and $a, b \in A$, we have $a \sim b$ so that $C = [a] = [b] = D$. Since the connected components cover A, and C is the only connected component which intersects with A, we have $A \subseteq C$.
We claim that the connected components are connected. Let C be a connected component. Let $a \in C$ so that $C = [a] = \{x \in X \mid x \sim a\}$. For each $x \in C$, we can choose a connected subspace $A_x \subseteq X$ with $a, x \in A_x$. By the previous claim, we have $A_x \subseteq C$ and have $C = \cup_{x \in C} A_x$. Since each A_x is connected and $a \in \cap_{x \in X} A_x$, it follows (from a previous theorem) that $\cup_{x \in X} A_x$ is connected and hence $C = \cup_{x \in X} A_x$ is connected. □

Definition 3.2.2 **Path Connectedness**

Let X be a topological space. For $a, b \in X$, a (continuous) *path* from a to b in X is a continuous map $\alpha : [0, 1] \subseteq \mathbb{R} \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. We say that X is *path-connected* when for every $a, b \in X$, there exists a path from a to b.

Theorem 3.9

Every path-connected space is connected.

Proof: Suppose X is path path connected. Suppose, for the sake of contradiction, that X is not connected. Choose open sets U, V in X which separate X. Let $a \in U, b \in V$. Since X is path-connected, we can choose a continuous map $\alpha : [0, 1] \subseteq \mathbb{R} \to X$ with $\alpha(0) = a, \alpha(1) = b$. Then the sets $\alpha^{-1}(U) \neq \varnothing$ and $\alpha^{-1}(V) \neq \varnothing$ are open in $[0, 1]$ and they separate $[0, 1]$. This contradicts the fact that intervals are connected. □

Theorem 3.10

The image of a path-connected space under a continuous map is path-connected. In particular, for topological spaces X and Y, if $X \cong Y$, then X is path-connected if and only if Y is path-connected.

Proof: Let $f : X \to Y$ be continuous and let X be path-connected. Let $c, d \in f(X)$. Choose $a, b \in X$ with $f(a) = c$ and $f(b) = d$. Since X is path-connected, we can choose a path α in X from a to b. Then $\beta = f \circ \alpha$ is a path in Y from c to d. □

Recall

For a subset A of a real vector space X, A is *convex* when for all $a, b \in A$, we have $[a, b] \subseteq A$ where $[a, b]$, where $[a, b]$ is the line segment in X,

$$[a, b] = \{a + t(b - a) \mid 0 \leq t \leq 1\}$$

Note

When X is a normed linear space (so it has a topology), every convex set in X is path-connected (because the map $\alpha : [0, 1] \subseteq \mathbb{R} \to X$ given by $\alpha(t) = a + t(b - a)$ is a path in X from a to b). More generally, the image of a convex set (in a normed linear space) under a continuous map is path-connected (hence connected).

Example

$A = \{x \in \mathbb{R}^2 \mid 1 \leq \|x\| \leq 2\}$ is the image of the (convex) rectangle $R = \{(r, \theta) \in \mathbb{R}^2 \mid r \in [1, 2], \theta \in [0, 2\pi]\}$ under the polar coordinates map $p : \mathbb{R}^2 \to \mathbb{R}^2$ given by $p(r, \theta) = (r \cos \theta, r \sin \theta)$.

Remark

In $\mathbb{R}^n$, open or closed rectangles and balls are convex. Also, open and closed balls in a normed linear space are convex.

Example

Using the product topology, a product of path connected spaces is path connected.
Proof: Let X_k be path connected for each $k \in K$. Let $a, b \in \prod_{k \in K} X_k$. For each $k \in K$, choose a path α_k in X_k from a_k to b_k. Then the map $\alpha : [0, 1] \to \prod_{k \in K} X_k$ given by $\alpha(t)(k) = \alpha(t)_k = \alpha_k(t)$ is a (continuous) path in $\prod_{k \in K} X_k$ from a to b. □

Example

Using the box topology, a product of path connected spaces need not be path connected. Using the box topology, $\mathbb{R}^\omega = \prod_{k=1}^{\infty} \mathbb{R}$ is not connected, and hence not path connected.

Definition 3.2.3

Let X be a topological space. Define a binary relation $\sim$ on X by stipulating that for $a, b \in X$,

$$a \sim b \Longleftrightarrow \text{there exists a path in } X \text{ from } a \text{ to } b$$

Note that this is an equivalence relation on X. For $a, b, c \in X$,

1. $a \sim a$ since the constant map ($\kappa_a : [0,1] \to X$ given by $\kappa_a(t) = t$ for all $t \in [0,1]$) is a path from a to a in X.
2. If $a \sim b$, then $b \sim a$ because if α is a path in X from a to b, then the map $\beta : [0,1] \subseteq \mathbb{R} \to X$ given by $\beta(t) = \alpha(1-t)$ is a path in X from b to a.
3. If $a \sim b$ and $b \sim c$, then we can choose a path α in X from a to b and a path β in X from b to c, and then $\gamma : [0,1] \subseteq \mathbb{R} \to X$ given by

$$\gamma(t) = \begin{cases} \alpha(2t) \text{ if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

   is a (continuous) path in X from a to c. γ is continuous by the Glueing Lemma.

The equivalence classes in X under $\sim$ are called the path-components of X. We note that X is the disjoint union of its path-components.

Theorem 3.11

Let X be a topological space. The path-components of X are the maximal path-connected subspaces of X. Indeed, each path-component of X is path-connected, and every path-connected subspace of X is contained in exactly one of the path-components of X.

Proof: Let P be a path-component of X. Let $a \in P$ so that $P = [a] = \{x \in X \mid x \sim a\}$. Then P is path-connected because given $b, c \in P$, we have $b \sim a$ and $c \sim a$, so that $b \sim c$ (so there is a path in X from a to b). Let A be any path-connected subspace of X. Let P and Q be any path-components for which $A \cap P \neq \varnothing$ and $A \cap Q \neq \varnothing$. Choose $p \in A \cap P$ and $q \in A \cap Q$. Since $p, q \in A$ and A is path-connected, we have $p \sim q$ and hence $P = [p] = [q] = Q$. Since the path-components cover X and A intersects with a unique path-component P, we have $A \subseteq P$. □

Note

In a topological space X, since each connected subspace of X is contained in (a unique) connected component of X, and since each path-component of X is path-connected, hence connected, it follows that each connected component of X is a (disjoint) union of some of the path-components of X.

Example **Topologist's Sine Curve**

Let $A = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : 0 < x \leq \frac{1}{\pi}\right\}, B = \{(0, y) : -1 \leq y \leq 1\}$ and let $X = A \cup B$. Verify that $\overline{A} = A \cup B = X$.
Sketch proof: $A \subseteq R = \left[0, \frac{1}{\pi}\right] \times [-1, 1] \subseteq \mathbb{R}^2$ and R is closed, so $\overline{A} \subseteq R$. Claim that $B \subseteq A'$. Let $p \in B$, say $p = (0, y)$ with $-1 \leq y \leq 1$. Let $\varepsilon > 0$. Choose $n \in \mathbb{Z}^+$ so that $\frac{2}{(2n+1)\pi} < \varepsilon$. For $a = \frac{2}{(2n+3)\pi}, b = \frac{2}{(2n+1)\pi}$, we have $0 < a < b < \varepsilon$ and one of $\sin\left(\frac{1}{a}\right)$ and $\sin\left(\frac{1}{b}\right)$ is equal to 1 and the other to -1. By IVT, we choose $x \in [a, b]$ so that $\sin\left(\frac{1}{x}\right) = y$ and then $\left(x, \sin\left(\frac{1}{x}\right)\right) \in A$ with $\left|\left(x, \sin\left(\frac{1}{x}\right) - (0, y)\right)\right| = x < b < \varepsilon$.
Now, since $B \subseteq A'$, $X = A \cup B \subseteq A \cup A' = \overline{A}$ and we have $\overline{A} \subseteq R$. To show that $\overline{A} = X$, it suffices to show that $(x, y) \in R, (x, y) \notin A \cup B \Longrightarrow (x, y) \notin A'$.
Let $(x, y) \in R$ with $(x, y) \notin A \cup B$. Since $(x, y) \notin B$, we have $x > 0$ and since $(x, y) \notin A$, $y \neq \sin\left(\frac{1}{x}\right)$. Let $\delta y = \left|y - \sin\left(\frac{1}{x}\right)\right|$ (choose $\delta > 0$ so that $\forall t \in \left(0, \frac{1}{n}\right], |t - x| < \delta \Longrightarrow \left|\sin\left(\frac{1}{t}\right) - \sin\left(\frac{1}{x}\right)\right| < \frac{\delta y}{2}$. So $\left|\sin\left(\frac{1}{t}\right) - y\right| \geq \left|y - \sin\left(\frac{1}{x}\right)\right| - \left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{t}\right)\right| > \frac{\delta y}{2}$. Then for $\varepsilon = \min\left(\delta, \frac{\delta y}{2}\right)$, we have $B((x, y), \varepsilon) \subseteq (x - \delta, x + \delta) \times \left(y - \frac{\delta y}{2}, y + \frac{\delta y}{2}\right)$. So that $B((x, y), \varepsilon) \cap A = \varnothing$. We have $X = A \cup B = \overline{A}$. Note that A is path-connected because it is the image of the convex set $\left(0, \frac{1}{\pi}\right]$ under the continuous map $g : \left(0, \frac{1}{\pi}\right] \to \mathbb{R}^2$ given by $g(x) = \left(x, \sin\left(\frac{1}{x}\right)\right)$. Also B is convex, hence path-connected. Note that X is connected because A is connected and $X = \overline{A}$. We claim that X is not path-connected, indeed there is no path from a point in A to a point in B. Since A and B are path connected with $\left(\frac{1}{\pi}, 0\right) \in A$ and $(0, 0) \in B$, it suffices to show that there is no path in $X = A \cup B$ from $\left(\frac{1}{\pi}, 0\right)$ to $(0, 0)$ (for $a \in A, b \in B$, if $a \sim b$, then since $\left(\frac{1}{\pi}, 0\right) \sim a, a \sim b, b \sim (0, 0)$ so that $\left(\frac{1}{\pi} 0\right) \sim (0, 0)$). Suppose, for a contradiction, that there is a path $\alpha : [0, 1] \to A \cup B$ from $\left(\frac{1}{\pi}, 0\right)$ to $(0, 0)$ in $X = A \cup B$. Note that the map $\alpha : [0, 1] \to \mathbb{R}^2$ is continuous, say α is given by $\alpha(t) = (x(t), y(t))$, where $x, y : [0, 1] \longrightarrow \mathbb{R}$ are both continuous with $(x(y), y(t)) \in X = A \cup B$ for all $t \in [0, 1]$ and with $x(0) = \frac{1}{\pi}, x(1) = 0, y(0) = y(1) = 0$. Also recall that when $(x, y) = X = A \cup B$ with $x > 0$, we have $(x, y) \in A$ so that $y = \sin\left(\frac{1}{x}\right)$. Since $x : [0, 1] \longrightarrow \mathbb{R}$ is continuous with $x(0) = \frac{1}{\pi}$ and $x(1) = 0$ by the IVT, we can choose $0 < t_1 < t_2 < \ldots < 1$ so that $x(t_n) = \frac{2}{(2n+1)\pi}$ and hence $y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = \sin\left((2n + 1)\frac{\pi}{2}\right) = (-1)^n$. Since $(t_n)_{n \geq 1}$ is increasing and bounded above (by 1), it converges with $\lim_{n \to \infty} t_n = s = \sup\{t_n \mid n \in \mathbb{Z}^+\} \leq 1$ and we have $0 < t_n < s \leq 1$ for all $n \in \mathbb{Z}^+$. Since $t_n \to s$ and since α is continuous at s, we have $(x(s), y(s)) = \alpha(s) = \lim_{n \to \infty} \alpha(t_n) = (\lim_{n \to \infty} x(t_n), \lim_{n \to \infty} y(t_n))$. We have $\lim_{n \to \infty} y(t_n) = \lim_{n \to \infty} (-1)^n$, which does not exist. So, in conclusion, $X = A \cup B$ is connected, but not path connected. Since X is connected, it only has one connected component, namely X. Since X is not path connected, it has at least two path components. So, since A and B are path connected with $X = A \cup B$, then A and B are the two path components of X.

Example

$\mathbb{R}$ is connected, so the only connected components of $\mathbb{R}$ is $\mathbb{R}$.
in $\mathbb{Q}$, the non-empty connected sets are the 1-point sets. The components of $\mathbb{Q}$ (and the path components of $\mathbb{Q}$) are the 1-point sets.
The intervals in $\mathbb{R}$ are equal to the connected sets in $\mathbb{R}$ (exercise). Note that the sets A in $\mathbb{R}$ with the intermediate value property are the convex sets in $\mathbb{R}$.

Example

Prove that for $A \subseteq \mathbb{R}$, A is convex (it satisifies the Intermediate Value Property) $\Longleftrightarrow$ A is connected.
If A is not convex, we can choose $a, bx \in \mathbb{R}$ with $a < x < b$ ad with $a, b \in A$ but $x \notin A$. But then the sets $(-\infty, x) \cap A$ and $(x, \infty) \cap A$ are (non-empty, disjoint) open sets in A which separate A. For the other direction, suppose that A is convex and suppose, for the sake of contradiction, that A is not connected. Choose open sets $U, V \subseteq A$ which separate A. Choose $a \in U, b \in V$. Since $U \cap V \neq \emptyset$, $a \neq b$, say $a < b$. Note that since A is convex, then $[a, b] \subseteq A$, and the set $U \cap [a, b]$ and $V \cap [a, b]$ are open sets in $[a, b]$ which separate $[a, b]$.

3.3 Compacrness

Definition 3.3.1 **Compactness**

Let X be a topological space. For a set $\mathcal{S}$ of subsets of X, we say $\mathcal{S}$ covers X (in X), or that $\mathcal{S}$ is a cover of X (in X), when $X = \cup\, \mathcal{S} = \cup_{A \in \mathcal{S}} A$.
When $\mathcal{S}$ is a cover of X (in X), a subcover of $\mathcal{S}$ (of X in X) is a subset $\mathcal{R} \subseteq \mathcal{S}$ such that $X = \cup\, \mathcal{R}$.
An open cover of X (in X) is a set of open sets in X which covers X.
We say that X is compact (in X) when every open cover of X is a finite subcover.
So X is compact (in itself) when for every set $\mathcal{S}$ of open sets in X with $X = \cup\, \mathcal{S}$, there exists a finite subset $\mathcal{R} \subseteq \mathcal{S}$ with $X = \cup\, \mathcal{R}$.

Theorem 3.12

The image of a compact set under a continuous map is compact. In particular, for topological spaces X and Y, if $X \cong Y$, then X is compact if and only if Y is compact.

Proof: Let $f : X \to Y$ be a function between topological spaces. Suppose that X is compact and that f is continuous. Note that (by restricting the codomain) the map $f : X \to f(X)$ is continuous. We claim that $f(X)$ is compact (in itself). Let $\mathcal{T}$ be an open cover of $f(X)$. Let $\mathcal{S} = \{f^{-1}(V) \mid V \in \mathcal{T}\}$. Then $\mathcal{S}$ is an open cover of X (each $f^{-1}(V)$ is open since f is continuous). Since X is compact, $\mathcal{S}$ has a finite subcover, so we can choose

$V_1, ... V_n \in \mathcal{T}$ so that $X = \cup_{k=1}^{n} f^{-1}(V_k)$. Then $f(X) = \cup_{k=1}^{n} V_k$ so that $\{V_1, ..., V_n\}$ is a finite subcover of $\mathcal{T}$. Thus $f(X)$ is compact, as claimed. □

Theorem 3.13 **Heine-Borel**

For $A \subseteq \mathbb{R}^n$, A is compact if and only if A is closed and bounded.

Definition 3.3.2

Let X be a subspace of Y. For a set $\mathcal{T}$ of subsets of Y, we say that $\mathcal{T}$ covers X in Y, or that $\mathcal{T}$ is a cover of X in Y, when $X \subseteq \cup \mathcal{T}$. When $\mathcal{T}$ is a cover of X in Y, a subcover of $\mathcal{T}$ (of X in Y) is a subset $\mathcal{R} \subseteq \mathcal{T}$ such that $X \subseteq \cup \mathcal{R}$. An open cover of X in Y is a set $\mathcal{T}$ of open sets in Y with $X \subseteq \cup \mathcal{T}$. We say that X is compact in Y when every open cover of X in Y has a finite subcover (of X in Y).

Theorem 3.14

Let X be a subspace of Y. Then X is compact (in itself) if and only if X is compact in Y.

Proof: Suppose X is compact (in X). Let $\mathcal{T}$ be an open cover of X in Y (the elements in $\mathcal{T}$ are open in Y and $X \subseteq \cup \mathcal{T}$). Let $S = \{V \cap X \mid V \in \mathcal{T}\}$. Note that S is an open cover of X (in X). Since X is compact (in X), we can choose $V_1, V_2, ..., V_n \in \mathcal{T}$ such that $X = \cup_{k=1}^{n} (V_k \cap X) = (\cup_{k=1}^{n} V_k) \cap X$. Then $X \subseteq \cup_{k=1}^{n} V_k$ so that $\{V_1, V_2, ..., V_n\}$ is a finite subcover of $\mathcal{T}$ (for X in Y). Suppose, conversely, that X is compact in Y. Let $\mathcal{S}$ be an open cover of X (in X). For each $U \in \mathcal{S}$ (so U is an open set in X), we choose V_U open in Y such that $U = V_U \cap X$. Then $\mathcal{T} = \{V_U \mid U \in \mathcal{S}\}$ is an open cover of X in Y. Since X is compact in Y, we can choose $U_1, ..., U_n \in \mathcal{S}$ such that $X \subseteq \cup_{k=1}^{n} V_{U_k}$. Then $X = \left(\cup_{k=1}^{n} V_{U_k}\right) \cap X = \cup_{k=1}^{n} \left(V_{U_k} \cap X\right) = \cup_{k=1}^{n} U_k$. So that $\{U_1, ..., U_n\}$ is a finite subcover of $\mathcal{S}$ (of X in X). □

Remark

When X is a subspace of a metric space Y (but not in general, when X is a subspace of a topological space), we have an analogous result for the connectedness of X in Y: X is connected in Y when there do not exist open sets U and V in Y which separate X in Y, meaning that $U \cap X \neq \varnothing, V \cap X \neq \varnothing, U \cap B = \varnothing, X \subseteq U \cup V$. Verify that (when Y is a metric space) X is connected (in itself) if and only if X is connected in Y.

Theorem 3.15

Every closed subspace of a compact space is compact

Proof: Let X be a subpace of Y. Suppose Y is compact (in Y) and that X is closed in Y. Let $\mathcal{S}$ be an open cover of X in Y. Since X is closed in Y, then $X^c = Y \setminus X$ is open in Y. Note that $\mathcal{S} \cup \{X^c\}$ is an open cover of Y (in Y). Since Y is compact, we can choose a finite subcover of $\mathcal{S} \cup \{X^c\}$. So we can choose a finite subset $\mathcal{R} \subseteq \mathcal{S}$ such that $\mathcal{R} \cup \{X^c\}$ covers Y. Then $\mathcal{R}$ is a finite subcover of $\mathcal{S}$ (of X in Y). □

Theorem 3.16

Every compact subspace of a Hausdorff space is closed.

Proof: Let $X \subseteq Y$ be a subspace. Suppose that X is compact and Y is Hausdorff. We shall prove that $X^c = Y \setminus X$ is open in Y. Let $b \in X^c$. For each $a \in X$, since Y is Hausdorff, we can choose disjoint open sets U_a and V_a in Y with $a \in U_a$ and $b \in V_a$. Note that $\mathcal{S} = \{U_a \mid a \in X\}$ is an open cover of X in Y. Since X is compact (in itself, hence also in Y), we can choose $a_1, \ldots, a_n \in X$ such that $X \subseteq \cup_{k=1}^n U_{a_k}$. Let $U = \cup_{k=1}^n U_{a_k}$ and let $V = \cap_{k=1}^n V_{a_k}$ (so $b \in V$). Note that $X \subseteq U$ and $U \cap V = \varnothing$ (since if we had $x \in U = \cup_{k=1}^n U_{a_k}$, we could choose k so that $x \in U_{a_k}$ and then $x \notin V_{a_k}$ (since $U_{a_k} \cap V_{a_k} \neq \varnothing$) so $x \notin \cap_{k=1}^n V_{U_k}$). Since $X \subseteq U$ and $U \cap V = \varnothing$, we also have $X \cap V = \varnothing$ so that $V \subseteq X^c$. We showed that for every $b \in X^c$, there exists an open set V in Y with $b \in V \subseteq X^c$. It follows that X^c is open in Y, hence X is closed in Y. □

Theorem 3.17

Let $f : X \to Y$ be a map between topological spaces. Suppose f is continuous and bijective, and X is compact and Y is Hausdorff. Then the inverse map $f^{-1} : Y \to X$ is continuous so that f is a homeomorphism.

Proof: Let $g = f^{-1} : Y \to X$. To show that g is continuous, we show that $g^{-1}(K)$ is closed in Y for every closed set $K \subseteq X$. Let K be a closed set in X. Since $g = f^{-1}$, we have $g^{-1}(K) = f(K)$. Since K is closed in X and X is compact, then K is compact. Since K is compact and f is continuous, then $f(K)$ is compact. Since $f(K)$ is a compact subspace of the Hausdorff space Y, $f(K)$ is closed in Y. Thus $g^{-1}(K) = f(K)$ is closed in Y for every closed set K in X, hence g is continuous. For $f : [0, 2\pi) \to \mathbb{S}^1$ given by $f(\theta) = e^{i\theta}$, f^{-1} is not continuous at $1 = (1, 0) \in \mathbb{S}^1$. □

Example

Recall that $\mathbb{R} \cong (0,1)$, $\mathbb{S}^1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$, $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$.
Moreover, $\mathbb{R} \ncong [0,1]$ since $[0,1]$ is compact, but $\mathbb{R}$ is not. Also, $\mathbb{R} \ncong [0,1)$ because $[0,1) \setminus \{0\} = (0,1)$ is connected, but $\mathbb{R} \setminus \{p\} = (-\infty, p) \cup (p, \infty)$ is not connected (if $f : [0,1) \to \mathbb{R}$ was a homeomorphism, and $f(0) = p$, then by restricting the domain and codomain for f and f^{-1}, the map $f : [0,1) \setminus \{0\} \to \mathbb{R} \setminus \{1\}$ would be a homeomorphism).
Further, no two of the spaces $\mathbb{R}^1, \mathbb{R}^2, \mathbb{S}^1, \mathbb{S}^2$ are homeomorphic. Indeed, $\mathbb{R}^1$ and $\mathbb{R}^2$ are not compact while $\mathbb{S}^1$ and $\mathbb{S}^2$ are compact, and $\mathbb{R}^1 \ncong \mathbb{R}^2$ because $\mathbb{R}^2 \setminus \{0\}$ is connected, but $\mathbb{R} \setminus \{p\}$ is not connected for any $p \in \mathbb{R}$, and $\mathbb{S}^1 \ncong \mathbb{S}^2$ because $\mathbb{S}^1 \setminus \{p\} \cong \mathbb{R}^1$ and $\mathbb{S}^2 \setminus \{q\} \cong \mathbb{R}^2$ using stereographic projection but $\mathbb{R}^1 \ncong \mathbb{R}^2$.

Example **Harder Examples**

$\mathbb{S}^2 \ncong \mathsf{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and more generally, $\mathbb{S}^n \ncong \mathsf{T}^n = \prod_{k=1}^n \mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$.
When $n, m \in \mathbb{Z}$ with $n \neq m$, $\mathbb{R}^n \ncong \mathbb{R}^m$, $\mathbb{S}^n \ncong \mathbb{S}^m$, $\mathsf{T}^n \ncong \mathsf{T}^m$. Note that T is a taurus.

Theorem 3.18

Let X be a topological space. Then X is compact if and only if X has the following property, which we call the finite intersection property on closed sets:
For every set $\mathcal{T}$ of closed subsets of X, if every finite subset of $\mathcal{T}$ has non-empty intersection, then $\cap \mathcal{T} \neq \varnothing$.

Proof: X is compact if and only if for all sets S of open sets in X with $X = \cup S$, there exists finite $R \subseteq S$ with $X = \cup R$, if and only if for all $\mathcal{T}$ of closed sets in X with $\cap T = \varnothing$, there exists finite $Q \subseteq \mathcal{T}$ with $\cap Q = \varnothing$ (by taking $\mathcal{T} = \{U^c \mid U \in S\}, S = \{K^c \mid K \in \mathcal{T}\}, Q = \{U^c \mid U \in R\}, R = \{K^c \mid K \in Q\}$), if and only if for all sets $\mathcal{T}$ of closed sets in X, if $\cap \mathcal{T} = \varnothing$, then there exists finite $Q \subseteq \mathcal{T}$ with $\cap Q = \varnothing$, if and only if for all $\mathcal{T}$ of closed sets in X if for all finite $Q \subseteq \mathcal{T}, \cap Q \neq \varnothing$, then $\cap T \neq \varnothing$. $\square$

Definition 3.3.3

A partially ordered set is a set X with a partial order $\leq$ which statisifes

$$\forall x, y \in X, x \leq x, (x \leq y \land y \leq x) \Rightarrow x = y, (x \leq y \land y \leq z) \Rightarrow x \leq z$$

A <u>Chain</u> in X is a subset $C \subseteq X$ such that $\forall x, y \in C \ (x \leq y \lor y \leq x)$. When $C \subseteq X$, an upper bound for C in X is an element $b \in X$ with $b \geq x$ for every $x \in C$.

Lemma 3.19 **Zorn's Lemma**

Let X be a partially ordered set. If every chain in X has an upper bound in X, then X has a maximal element ($\exists a \in X, \nexists x \in X, a < x$).

Theorem 3.20 **Tychonoff's Theorem**

The product of a set of compact spaces is compact, using the product topology.

Proof: Let K be a non-empty set, and let X_k be a compact topological space for each $k \in K$. To prove that $\prod_{k\in K} X_k$ is compact, we shall show that it has the finite intersection property on closed subsets. Let T be any set of closed subsets of $\prod_{k\in K} X_k$ such that for every finite $R \subseteq T, \cap R \neq \varnothing$. We must show that $\cap T \neq \varnothing$. By Zorn's Lemma, there exists a maximal set S of (not necessarily closed) subsets of $\prod_{k\in K} X_k$, which contains T, such that intersections of finite subsets of S are not empty. Note that by the maximality of S, S is closed under finite intersection (if $A_1, ..., A_n \in S$, then $S \cup \{A_1 \cap ... \cap A_n\}$ has the property that intersections of finite subsets are non-empty). We shall show that $\cap \left\{\overline{A} \mid A \in S\right\} \neq \varnothing$ and hence $\cap T \neq \varnothing$ because $T \subseteq S$ and $A \in T \Rightarrow A$ is closed $\Rightarrow A = \overline{A}$ so that $T \subseteq \left\{\overline{A} \mid A \in S\right\}$. For each index $k \in K$, let $p_l : \prod_{k\in K} X_k \to X_l$ be the lth projection map. Consider $\{p_l(A) \mid A \in S\}$. Since finite subsets of $S = \{A \mid A \in S\}$ have non-empty intersection, therefore, finite subsets of $\left\{p_p(A) \mid A \in S\right\}$ have non-empty intersection, therefore, finite subsets of $\left\{\overline{p_l(A)} \mid A \in S\right\}$ have non-empty intersection. Since X_l is compact and finite subsets of $\left\{\overline{p_l(A)} \mid A \in S\right\}$ have non-empty interection, it follows that $\cap \left\{\overline{p_l(A)} \mid A \in S\right\} \neq \varnothing$. So we can choose $a_k \in X_k$ so that $a_k \in \overline{p_k(A)}$ for every $A \in S$. We do this for each $k \in K$ to get an element $a \in \prod_{k\in K} X_k$ such that $\forall k \in K, \forall A \in S, a_k \in \overline{p_k(A)}$. We shall prove that $a \in \overline{A}$ for every $A \in S$. It suffices to show that $U \cap A \neq \varnothing$ for every basic open set U in $\prod_{k\in K} X_k$ with $a \in U$. Let U be a basic open set in $\prod_{k\in K} X_k$ with $a \in U$. Say $U = \prod_{k\in K} U_k$, where each U_k is open in X_k with $a_k \in U_k$ and with $U_k = X_k$ for all $k \notin J$, where J is a finite subset of K. For each $k \in K$, for each $A \in S$, $a_k \in \overline{p_k(A)}$, so $p_k(A) \cap U_k \neq \varnothing$, hence $A \cap p_k^{-1}(U_k) \neq \varnothing$. Since $p_k^{-1}(U_k) \cap A \neq \varnothing$ for every $A \in S$, we have $p_k^{-1}(U_k) \in S$ (if $p_k^{-1}(U_k) \notin S$, then $S \not\subset S \cup \{p_k^{-1}(U_k)\}$ but $S \cup \{p_k^{-1}(U_k)\}$ satisfies (1) and (2)). Then we have

$U = \prod_{k\in K} U_k = \left\{x \in \prod_{k\in K} X_k \mid x_k \in U_k \forall k \in K\right\} = \left\{x \in \prod_{k\in K} X_k \mid x_k \in U_k \forall k \in J\right\} = \left\{x \in \prod_{k\in K} X_k \mid p_k(x) \in U_k \forall k \in J\right\} = \left\{x \in \prod_{k\in K} X_k \mid x \in p_k^{-1}(U_k) \forall k \in J\right\} = \cap_{k\in J}\, p_k^{-1}(U_k)$.

Since each $p_k^{-1}(U_k) \in S$, then $U = \cap_{k\in J}\, p_k^{-1}(U_k) \in S$ (if $A_1, ..., A_n \in S$ and $A_1 \cap ... \cap A_n \neq S$, then $S \not\subset S \cup \{A_1 \cap ... \cap A_n\}$ but $S \cup \{A_1 \cap ... \cap A_n\}$ satisfies (1) and (2)). Since $U \in S$, we have $U \cap A \neq \varnothing$ for every $A \in S$ (since S satisfies (1)). This shows that for every A in S, for every basic open set U in $\prod_{k\in K} X_k$ with $a \in U$, we have $U \cap A \neq \varnothing$ and hence $a \in \overline{A}$. Thus we have $a \in \overline{A}$ for all $A \in S$, so $a \in \cap \left\{\overline{A} \mid A \in S\right\}$, hence $a \in \cap T$, since $T \subseteq \left\{\overline{A} \mid A \in S\right\}$. □

Example

$[0,1]^\omega$ is compact in $\mathbb{R}^\omega$, using the product topology.
Show that $[0,1]^\omega$ is not compact when $\mathbb{R}^\omega$ uses the box topology.

4 Countability and Separation Axioms

Definition 4.0.1

Let X be a topological space.

1. We say that X is first countable when for each $a \in X$, there exists countable (finite or infinitely countable) set $\mathcal{B}$ of open sets in X such that for every open U in X with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
2. We say that X is second-countable when there exists a countable basis $\mathcal{B}$ for the topology on X.
3. We say that X is Lindeloff when every open cover of X has a countable subcover. We say that X is separable when there exists a countable dense subset A in X.

Theorem 4.1

1. Every metric space is first-countable.
2. For every metric space X, X is second-countable $\iff$ X is Lindeloff $\iff$ X is separable.

Theorem 4.2

Let X be a topological space. If X is second-countable, then X is first-countable, X is Lindeloff and X is separable.

Proof: Suppose X is 2nd countable, and let $\mathcal{B}$ be a countable basis for the topology on X. It is clear that X is first countable (for each $a \in X$, let $\mathcal{B}_a = \mathcal{B}$). To show that X is Lindeloff, let $\mathcal{S}$ be an open cover of X. For each $a \in X$, choose $U_a \in \mathcal{S}$ with $a \in U_a$, then choose $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U_a$. Then $\{B_a \mid a \in X\}$ is an open cover for X and it is countable since $\{B_a \mid a \in X\} \subseteq \mathcal{B}$. Choose $a_1, a_2, \ldots$ in X such that $\{B_a \mid a \in X\} = \left\{B_{a_1}, B_{a_2}, \ldots\right\}$. Then $\left\{U_{a_1}, U_{a_2}, \ldots\right\}$ is a countable subcover of X (since given $a \in X$, we have $a \in B_a$ and we can choose k so that $B_a = B_{a_k}$ and then $a \in B_{a_k} \subseteq U_{a_k}$). To prove that X is separable, write $\mathcal{B} = \{B_1, B_2, \ldots\}$. For each $k \geq 1$, choose $a_k \in B_k$, then let $A = \{a_1, a_2, \ldots\}$. Then A is dense in X because for each $k \geq 1$, $a_k \in A \cap B_k$ so that $A \cap B_k \neq \varnothing$ (when X is a topological space and $A \subseteq X$, A is dense in X if and only if $\overline{A} = X$ iff for every $a \in X$, $a \in \overline{A}$ iff for every $a \in X$, for open set U in X with $a \in U$, we have $A \cap U \neq \varnothing$ iff for every basic open set $B \in \mathcal{B}$ (where $\mathcal{B}$ is any basis for X), we have $A \cap B \neq \varnothing$). $\square$

Warning

When X is not second-countable, the other 3 properties do not imply one another.

Example

	1st countable	Lindolff	Separable	2nd countable
$\mathbb{R}_l$	✓	✓	✓	✗
I_0^2	✓	✓	✗	✗
Γ	✓	✗	✓	✗
$\mathbb{R}_{\text{cf}}$	✗	✓	✓	✗
$\mathbb{R}_{\text{cc}}$	✗	✓	✗	✗
$\mathbb{R}_{\text{d}}$	✓	✗	✗	✗
exercise	✗	✗	✓	✗

I_0^2 denotes the topological space using the set $I^2 = [0,1] \times [0,1]$ and using the dictionary order topology.
Γ (possibly called the Moore plane) is the topological space which uses the set $\mathbb{R}^2$, whose underlying set is the closed upper half plane using basic open sets of the form

$$B((a,b),r), 0 < r < b$$

and of the form

$$(B((a,0),r) \cap \Gamma) \cup \{(a,0)\}$$

$\mathbb{R}_{\text{cf}}$ is the set $\mathbb{R}$ using the cofinite topology (the closed set are the finite sets and $\mathbb{R}$).

Theorem 4.3

1. Every subspace of a first countable space is first countable
2. Every subspace of a second countable space is second countable.

Proof: Exercise. □

Theorem 4.4

1. The product of any two first countable spaces is first countable.
2. The product of any two second countable spaces is second countable.
3. The product of any two separable spaces is separable.

Proof: Exercise. □

Example

The subspace of a Lindeloff space is not necessarily Lindeloff.
A subspace of a separable space is not necessarily separable.
The product of two Lindeloff spaces is not necessarily Lindeloff.

Definition 4.0.2 **Separation Axioms**

Let X be a topological space.

1. We say X is $T1$ or that the 1-point subsets of X are closed when the 1-point subsets of X are closed in X.
2. We say that X is $T2$, or that X is Hausdorff, when for all $a, b \in X$ with $a \neq b$, there exist disjoint open U and V in X with $a \in U$ and $b \in V$.
3. We say that X is $T3$, or that X is regular, when 1-point subsets of X are closed, and for all $a \in X$, and for every closed set B in X with $a \notin B$, there exist disjoint open sets U and V in X with $a \in U$ and $B \subseteq V$.
4. We say that X is $T4$, or that X is normal, when 1-point subsets of X are closed and for any two disjoint closed sets A and B in X, there exist disjoint open sets U and V in X with $A \subseteq U$ and $B \subseteq V$.
5. We say that X is metrizable when there exists a metric on X for which the metric topology is equal to the topology on X.

Theorem 4.5

Let X be a topological space.

1. If X is metrizable, then X is normal.
2. If X is normal, then X is regular.
3. If X is regular, then X is Hausdoff.
4. If X is Hausdorff, then the 1-point subsets of X are closed.

Proof: An exercise. □

Example **Showing that Separation Axioms differ in Strength**

Not every space X in which the 1-point subsets are closed is Hausdorff:
for example, let X be any infinite set with the cofinite topology.
Not every Hausdorff space is regular:
for example, let $K = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$ and let $\mathbb{R}_k$ be the topological space whose underlying set is $\mathbb{R}$ with basis consisting of sets of the forms (a, b) with $a < b$ and $(a, b) \setminus K$ with $a < b$. This space is Hausdorff but not regular because teh point 0 cannot be separated from the closed set K.
Not every regular space is normal.
for example, verify that

1. $\mathbb{R}_l$ is normal (hence regular)
2. the product of two regular spaces is regular

so that, in particular, $\mathbb{R}_l \times \mathbb{R}_l$ is regular, but

1. $\mathbb{R}_l \times \mathbb{R}_l$ is not normal.

Proof:

1. It is clear that $\mathbb{R}_l$ is Hausdorff. Let A, B be disjoint closed sets in $\mathbb{R}_l$. For each $a \in A$, since $B^c = \mathbb{R} \setminus B$ is open with $a \in B^c$, we can choose $r_a > a$ so that $[a, r_a) \cap B = \varnothing$. For each $b \in B$, choose $s_b > b$ so that $[b, s_b) \cap A = \varnothing$. Then the sets $U = \cup_{a \in A} [a, r_a)$ and $V = \cup_{b \in B} [b, s_b)$ are disjoint open sets in $\mathbb{R}_l$ with $A \subseteq U$ and $B \subseteq V$.
2. Exercise.
3. Consider the diagonal $L = \{(x_1 - x) \mid x \in \mathbb{R}\}$. Note that the subspace topology of L is the discrete topology. Also, note that L is closed in $\mathbb{R}_l \times \mathbb{R}_l$. Thus, every subset $A \subseteq L$ is closed in $\mathbb{R}_l \times \mathbb{R}_l$. Suppose, for a contradiction, that $\mathbb{R}_l \times \mathbb{R}_l$ is normal. Then given any $A \subseteq L$, we can find disjoint open sets U_A and V_A in $\mathbb{R}_l \times \mathbb{R}_l$ with $A \subseteq U_A$ and $L \setminus A \subseteq V_A$, and to be specific, choose, $U_\varnothing = \varnothing, V_\varnothing = \mathbb{R}_l \times \mathbb{R}_l, U_L = \mathbb{R}_l \times \mathbb{R}_l, V_L = \varnothing$. Define $F : P(L) \to P(\mathbb{Q}^2)$ by $F(A) = U_A \cap \mathbb{Q}^2$. Note that F is injective. Thus $F(A) \neq F(B)$.

□

Note

Verify that all of the countability properties and separation properties in this chapter are invariant under homeomorphism (if $X \cong Y$, then X has one of the properties if and only if Y has the same property).

Example

$\mathbb{R}_l$ is not second countable.
Proof: let B be any basis for $\mathbb{R}_l$. For each $a \in \mathbb{R}_l$, choose a basic open set $B_a \in B$ with $a \in B_a \subseteq [a, a+1)$ and note that $a = \min B_a$. Define $F : \mathbb{R}_l \to B$ by $F(a) = B_a$. Then F is injective since if $F(a) = F(b)$, then $B_a = B_b$. So $a = \min B_a = \min B_b = b$. Since F is injective, then $2^{\aleph_0} = |\mathbb{R}_l| \le |B|$, so that B is uncountable. □

Theorem 4.6

Every subspace of a $T1$ space is $T1$, every subspace of a Hausdorff is Hausdorff, every subspace of a regular space is regular, and every subspace of a metrizable space is metrizable.

Proof:

1. Let $X \subseteq Y$ be a subspace. Suppose Y is $T1$. For $a \in X$, $\{a\}$ is closed in Y and $\{a\} = \{a\} \cap X$, so it is closed in X.
2. Let $a, b \in X$ with $a \neq b$. Since Y is Hausdorff, we can choose disjoint open sets U and V in Y with $a \in U$ and $b \in V$. Then $U \cap X$ and $V \cap X$ are disjoint open sets in X with $a \in U \cap X$ and $b \in V \cap X$.
3. Suppose Y is regular. Since Y is $T1$, so is X. Let $a \in X$ and B be a closed set with $a \notin B$. Note that since B is closed in X, $B = Cl_X(B) = \overline{B} \cap X$, where $\overline{B} = Cl_Y(B)$. Since $a \in X$ and $a \notin B = \overline{B} \cap X$, we have $a \notin \overline{B}$. Since Y is regular, we can choose disjoint open sets U and V in Y with $a \in U$ and $\overline{B} \subseteq V$. Then $U \cap X$ and $V \cap X$ are disjoint open sets in X with $a \in U \cap X$ and $B = \overline{B} \cap X \subseteq V \cap X$.
4. The proof that subspaces of metrizable spaces are metrizaible is an **exercise**.

□

Remark

A subspace of a normal space need not be normal. We shall soon see that for any set K, $[0,1]^k$ is normal. It can (maybe) be shown that $(0,1)^k \subseteq [0,1]^k$ is not normal when K is uncountable.

Theorem 4.7 **Alternative Definition of Regularity**

Let X be a $T1$ space. Then X is regular if and only if for every $a \in X$ and every open set W in X with $a \in W$, there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$.

Proof: ($\Longrightarrow$)Suppose X is regular. Let W be an open set in X and let $a \in W$. Then W^c is closed and $a \notin W^c$, so, since X is regular, we can choose disjoint open sets U and V in X with $a \in U$ and $W^c \subseteq V$. Since $U \cap V = \varnothing$, we have $U \subseteq V^c$ which is closed, so $\overline{U} \subseteq V^c$

and since $W^c \subseteq V$, we have $V^c \subseteq W$, so we have $a \in U \subseteq \overline{U} \subseteq V^c \subseteq W$.
$(\Longleftarrow)$ Suppose, conversely, that for every open W in X and every $a \in W$, there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$. Let $a \in X$ and let B be a closed set in X with $a \notin B$. Taking $W = B^c$, we choose an open set U in X with $a \in U \subseteq \overline{U} \subseteq B^c$. Let $V = \left(\overline{U}\right)^c$, which is open. Since $\overline{U} \subseteq B^c$, we have $B \subseteq \left(\overline{U}\right)^c = V$, and since $U \subseteq \overline{U}$, we have $U \cap \overline{U}^c = \varnothing$, that is $U \cap V = \varnothing$. □

Theorem 4.8

For each $k \in K$, let X_k be a topological space and let $A_k \subseteq X_k$. Then using either the product or the box topology,

$$\overline{\prod_{k \in K} A_k} = \prod_{k \in K} \overline{A_k}$$

Proof: For each $l \in K$, let $p_l : \prod_{k \in K} X_k \to X_k$ be the $l'th$ projection map. Since p_l is continuous and $\overline{A}_l$ is closed in X_l, $p_l^{-1}\left(\overline{A}_l\right) = \left\{x \in \prod_{k \in K} X_k \mid x_k \in \overline{A}_k\right\}$ is closed in $\prod_{k \in K} X_k$. It follows that

$$\begin{aligned} \prod_{k \in K} \overline{A_k} &= \left\{x \in \prod_{k \in K} X_k \mid x_k \in \overline{A}_k \forall k \in K\right\} \\ &= \bigcap_{k \in K} p_k^{-1}\left(\overline{A}_k\right) \end{aligned}$$

is closed in $\prod_{k \in K} X_k$. Since $\prod_{k \in K} A_k \subseteq \prod_{k \in K} \overline{A_k}$ and $\prod_{k \in K} \overline{A_k}$ is closed, we have $\overline{\prod_{k \in K} A_k} \subseteq \prod_{k \in K} \overline{A_k}$. Let $b \in \prod_{k \in K} \overline{A_k}$. To show that $b \in \overline{\prod_{k \in K} A_k}$, it suffices to show that for every open set W in $\prod_{k \in K} X_k$ with $b \in W$, we have $\prod_{k \in K} A_k \cap W \neq \varnothing$. Let W be any open set in $\prod_{k \in K} X_k$ with $b \in W$. Choose a basic open set U in $\prod_{k \in K} X_k$ with $b \in U$, say $U = \prod_{k \in K} U_k$, where each U_k is open in X_k with $b_k \in U_k$. For each $k \in K$, since $b \in \prod \overline{A_k}$, so $b_k \in \overline{A_k}$, we have $A_k \cap U_k \neq \varnothing$. Since $A_k \cap U_k \neq \varnothing$ for all $k \in K$, we have $\prod_{k \in K} A_k \cap U = \prod_{k \in K} A_k \cap \prod_{k \in K} U_k = \prod_{k \in K} (A_k \cap U_k) \neq \varnothing$. □

Theorem 4.9

Using either the product or box topology, every product of T1 spaces is T1, every product of Hausdorff spaces is Hausdorff, and every product of regular spaces is regular.

Proof: Let X_k be a topological space for each $k \in K$. Suppose each X_k is $T1$. Note that for $a \in \prod_{k \in K} X_k$, for each $k \in K$, since X_k is $T1$, $\overline{\{a_k\}} = \{a_k\}$ in X_k. Also note that for $a \in \prod_{k \in K} X_k$,
$\{a\} = \{x \in \prod X_k \mid x_k = a_k \forall k \in K\} = \left\{x \in \prod_{k \in K} X_k \mid x_k \in \{a_k\} \forall k \in K\right\}$
$= \prod_{k \in K} \{a_k\}$
. So in $\prod_{k \in K} X_k$, $\overline{\{a\}} = \overline{\prod_{k \in K} \{a_k\}} = \prod_{k \in K} \overline{\{a_k\}} = \prod_{k \in K} \{a_k\} = \{a\}$. Now, suppose that each X_k is Hausdorff. Let $a, b \in \prod_{k \in K} X_k$ with $a \neq b$. Since $a \neq b$, we can choose $l \in K$ such

that $a_l \neq b_l$. Since X_k is Hausdorff, we can choose disjoint open sets U_l and V_l in X_l with $a_l \in U_l$ and $b_l \in V_l$. Let $U_k = V_k = X_k$ for all $k \in K$ with $k \neq l$. Then $U = \prod_{k\in K} U_k$ and $V = \prod_{k\in K} V_k$ are disjoint open sets in $\prod_{k\in K}$ and $a \in U$ and $b \in V$. Now, suppose each X_l is regular (and hence $T1$, so that $\prod_{k\in K} X_k$ is $T1$). To show that $\prod_{k\in K} X_k$ is regular, we show that for every open W in $\prod_{k\in K} X_k$ and every $a \in W$, there exists an open set U in W with $a \in U \subseteq \overline{U} \subseteq W$. Let $a \in \prod_{k\in K} X_k$ and let W be an open set in $\prod_{k\in K} X_k$ with $a \in W$. Choose a basic open set V in $\prod_{k\in K} X_k$ with $a \in V \subseteq W$, say $V = \prod_{k\in K} V_k$, where each V_k is open in X_k with $a_k \in V_k$. For each $k \in K$, since X_k is regular, we can choose an open set U_k in X_k with $a \in U_k \subseteq \overline{U_k} \subseteq V_k$, choosing $U_k = X_k$ whenever $V_k = X_k$, then for $U = \prod_{k\in K} U_k$, we have

$$a \in U \subseteq \overline{U} = \overline{\prod_{k\in K} U_k} = \prod_{k\in K} \overline{U_k} \subseteq \prod_{k\in K} V_k = V \subseteq W$$

□

Remark

Products of normal spaces are not necessarily normal (for example, $\mathbb{R}_l$ is normal, but $\mathbb{R}_l \times \mathbb{R}_l$ is not).

Theorem 4.10

1. Every compact Hausdorff space is normal.
2. Every regular, second countable space is normal.
3. Every ordered set (with maximum or a minimum element), using the order topology, is normal.

Proof: Exercise. □

Theorem 4.11 **Urysohn's Lemma**

Let X be a normal topological space and let A and B be disjoint closed sets. Then there exists a continuous map $f : X \to [0, 1] \subseteq \mathbb{R}$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Proof: Snew Ch. 4 □

Theorem 4.12 **Urysohn's Metrization Theorem**

Every regular, second-countable topological space is metrizable.

Proof: Snew Ch. 4 □

5 Topological Manifolds

Definition 5.0.1

A topological space X is **locally homeomorphic** to $\mathbb{R}^n$ when X has an open cover S such that every $U \in S$ is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 5.0.2

An **n-dimensional topological manifold**, also called a (topological) **n-manifold**, is a Hausdorff, second-countable topological space X which is locally homeomorphic to $\mathbb{R}^n$.

Definition 5.0.3

When X is an n-manifold, for each $a \in X$, we can choose an open set U_a in X with $a \in U_a$ and a homeomorphism $\varphi_a : U_a \subseteq X \to \varphi_a(U_a) \subseteq \mathbb{R}^n$ from U_a (which is open in X) to $\varphi_a(U_a)$ (which is open in $\mathbb{R}^n$). $S = \{U_a \mid a \in X\}$ is an open cover of X. When $S = \{U_k \mid k \in K\}$ is an open cover of X and for each $k \in K$, we have a homeomorphism $\varphi_k : U_k \subseteq X \to \varphi_k(U_k) \subseteq \mathbb{R}^n$ from U_k to an open set $\varphi_k(U_k)$ in $\mathbb{R}^n$, the sets U_k are called **coordinate neighbourhoods** and the homeomorphisms $\varphi_k : U_k \to \varphi_k(U_k)$ are called (coordinate) **charts**, and the set $\mathcal{A} = \{\varphi_k \mid k \in K\}$ is called an **atlas** for X. The homeomorphisms $\varphi_k \varphi_l^{-1} : \varphi_l(U_k \cap U_l) \to \varphi_k(U_k \cap U_l)$ are called **transition maps**, or **change of coordinate maps**.

Example

$\mathbb{R}^n$ is an n-manifold. We can use one chart $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ where φ is the identity map.

Example

Any n-dimensional real vector space X is an n-manifold: choose a basis $\{u_1, ..., u_n\}$ for X, and define $\varphi : X \to \mathbb{R}^n$ by $\varphi\left(\sum_{i=1}^n t_i u_i\right) = t$.

Example

Any open subset of an n-manifold is an n-manifold. For example, the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices is an n^2-manifold and $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is an open set in $M_n(\mathbb{R})$. The **n-sphere** $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is a compact n-manifold: we can use two charts. Let $p = e_{n+1} = (0, ..., 0, 1)$. Then the stereographic projections $\varphi : \mathbb{S}^n \setminus \{p\} \to \mathbb{R}^n$ and $\psi : \mathbb{S}^n \setminus \{-p\} \to \mathbb{R}^n$ are homeomorphisms.

Example

On $\mathbb{S}^n$, we could also let $U_k = \{x \in \mathbb{S}^n \mid x_k > 0\}$, $V_k = \{x \in \mathbb{S}^n \mid x_k < 0\}$ and define

$$\varphi_k : U_k \to B(0,1) \subseteq \mathbb{R}^n$$

and

$$\psi_k : V_k \to B(0,1) \subseteq \mathbb{R}^n$$

by

$$\begin{aligned}\varphi_k(x_1, ..., x_{n+1}) &= (x_1, ..., x_k, ..., x_{n+1}) \\ &= (x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+1})\end{aligned}$$

and

$$\psi_k(x_1, ..., x_{n+1}) = (x_1, ..., x_k, ..., x_{n+1})$$

$$\varphi_k^{-1}(y_1, ..., y_n) = \left(y_1, ..., y_{k-1}, \sqrt{1 - \|y\|^2}\right), y_k, ..., y_n$$

Example

When X is an n-manifold and Y is an m-manifold, the product $X \times Y$ is an $(n+m)$-manifold.
For example, the ***n*-torus** $T^n = \prod_{k=1}^n \mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{S}^1 \times ... \times \mathbb{S}^1$ is an n-manifold.

Example **Real Projective n-Space**

For a real vector space W, the **projectivization** of W is the set $P(W)$ of 1-dimensional subspaces of W,

$$\mathbb{P}(W) = \{[x] \mid 0 \neq x \in W\}$$

where $[x] = \operatorname{span}_{\mathbb{R}}(\{x\})$.
An element in $\mathbb{P}(W)$ is called a **point** in $P(W)$, a **line** in $\mathbb{P}(W)$ is a set of the form $\mathbb{P}(U)$ for some 2-dimensional subspace U in W, and a **plane** in $\mathbb{P}(W)$ is a set of the form $\mathbb{P}(U)$ for some 3-dimensional subspace of W.
The n-dimensional **real projective space** is the set

$$\mathbb{P}^n = \mathbb{P}^n(\mathbb{R}) = \mathbb{P}(\mathbb{R}^{n+1}) = \{[x] \mid 0 \neq x \in \mathbb{R}^{n+1}\}$$

where $[x] = \operatorname{span}_{\mathbb{R}}(\{x\})$. We give $\mathbb{P}^{n(\mathbb{R})}$ a topology by identifying it with the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$. Under this identification,

$$\mathbb{P}^n = \mathbb{P}^n(\mathbb{R}) = \{[x] \mid 0 \neq x \in \mathbb{R}^{n+1}\}$$

where $[x] = \{tx \mid t \in \mathbb{R}^*\} = \operatorname{span}_{\mathbb{R}}(\{x\}) \setminus \{0\}$

Exercise 5.0.1

Show that $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^* \cong \mathbb{S}^n/\{\pm 1\}$.

Note

$\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ is an n-manifold.

Remark

$\mathbb{P}^2$ is is not orientable.

Example **of Non-Manifolds**

1. Let $X = \mathbb{R} \cup \{p\}$, where $p \notin \mathbb{R}$, with the topology generated by the basis consisting of sets $(a, b) \subseteq \mathbb{R}$ with $a, b \in \mathbb{R}$, $a < b$, and $(a, 0) \cup \{p\} \cup (0, b)$ where $a < 0 < b$. Then X is locally homeomoprhic to $\mathbb{R}^1$ (and X is second-countable), but X is not Hausdorff (since 0 and p cannot be seaparated by disjoint open sets).
2. There are well-ordered sets, called **ordinals**, such that every well-ordered set is isomorphic to a unique ordinal. The first "few" ordinals are $0 = \varnothing, 1 = \{0\}, 2 = \{0, 1\}, ..., n + 1 = n \cup \{n\}, ..., \omega = \mathbb{N} = \{0, 1, ...\}, \omega + .$ $1 = \omega \cup \{\omega\}, ..., \Omega = \aleph_1$
   There is a first uncountable ordinal Ω. We give the set $\Omega \times [0, 1)$ the dictionary order topology, and the **long line** is the subspace
   $$L = (\Omega \times [0, 1)) \setminus \{(0, 0)\}$$
   L is locally homeomorphic to $\mathbb{R}^1$ (and is Hausdorff), but is not 2nd countable.

Exercise 5.0.2

Show that every Hausdorff space which is locally homeomrphic to $\mathbb{R}^n$ is regular (hence also metrizable).

Remark

It can be shown that every compact n-manifold is homeomorphic to a subspace of $\mathbb{R}^m$ for some $m \in \mathbb{Z}^+$.

On IPad.

6 Fundamental Group

6.1 The Fundamental Group

Recall

Let X be a topological space with $a, b \in X$. Recall that a path from a to b in X is a continuous map $\alpha : [0, 1] \subseteq \mathbb{R} \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Recall that a **loop** at a in X is a path from a to a in X.

Definition 6.1.1

Given two paths α and β from a to b in X, an (endpoint-fixing) homotopy from α to β in X is a continuous map $F : [0,1] \times [0,1] \to X$ such that $F(0,t) = \alpha(t)$ and $F(1,t) = \beta(t)$ for all $t \in [0,1]$ and $F(s,0) = a$ and $F(s,1) = b$ for all $s \in [0,1]$. We say that **α is homotopic to β** in X and we write $\alpha \sim \beta$, when there exists a homotopy from α to β in X.

Theorem 6.1

Homotopy equivalence is an equivalence relation on the set of all paths from a to b in a topological space X.

Proof: Let α, β and γ be paths from a to b in X. Then $\alpha \sim \alpha$, indeed the map $F : [0,1] \times [0,1] \to X$ given by $F(s,t) = \alpha(t)$ for all $s \in [0,1]$ is a homotopy from α to β in X.
If $\alpha \sim \beta$, then $\beta \sim \alpha$, indeed if $F : [0,1] \times [0,1] \to X$ is a homotopy, from α to β in X, then $G : [0,1] \times [0,1] \to X$ given by $G(s,t) = F(1-s,t)$ is a homotopy from β to α in X.
Finally, if $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$: indeed, if F is a homotopy from α to β in X and G is a homotopy from β to γ in X, then the map $H : [0,1] \times [0,1] \to X$ given by

$$H(s,t) = \begin{cases} F(2s,t) \text{ for } 0 \leq s \leq \frac{1}{2}, 0 \leq t < 1 \\ G(2s-1,t) \text{ for } \frac{1}{2} \leq s \leq 1, 0 \leq t \leq 1 \end{cases}$$

□

Definition 6.1.2

For a topological space X and a point $a \in X$, we let $\Pi_1(X,a)$ be the set of equivalence classes of loops at a in X under homotopy equivalence:

$$\Pi_1(X,a) = \{[a] \mid \alpha \text{ is a loop at } a \text{ in } X\}$$

where

$$[\alpha] = \{\beta \mid \beta \text{ is a loop at } a \text{ in } X \text{ with } \beta \sim \alpha\}$$

Definition 6.1.3

For a topological space X with $a, b, c \in X$, the **constant loop** κ_a at a in X is the loop $\kappa_a : [0,1] \to X$ given by $\kappa_a(t) = a$ for all $t \in [0,1]$. For a path α in X from a to b, the **inverse path** α^{-1} in X from b to a is the map $\alpha^{-1} : [0,1] \to X$ given by $(\alpha^{-1})(t) = \alpha(1-t)$. For a path α in X from a to b, and a path β in X from b to c, the **product path** $\alpha\beta$ in X from a to c is the map $\alpha\beta : [0,1] \to X$ given by

$$(\alpha\beta)(t) = \begin{cases} \alpha(2t), 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), \frac{1}{2} \leq t \leq 1 \end{cases}$$

Theorem 6.2

For a topological space X with $a \in X$, $\pi_1(X, a)$ is a group with identity $1 = [\kappa_a]$ and with $[\alpha]^{-1} = [\alpha^{-1}]$ and with $[\alpha][\beta] = [\alpha\beta]$. This group is called the **fundamental group**, (or the **first homotopy group** of X at a).
Indeed, we have the following:

1. For paths α and β from a to b in X, if $\alpha \sim \beta$ in X, then $\alpha^{-1} \sim \beta^{-1}$ in X.
2. For paths α and β from a to b in X and for paths γ and δ from b to c in X, if $\alpha \sim \beta$ in X and $\gamma \sim \delta$ in X, then $\alpha\gamma \sim \beta\delta$ in X.
3. For a path α from a to b in X, $\kappa_a\alpha \sim \alpha$ in X, $\alpha\kappa_b \sim \alpha$ in X.
4. For a path α from a to b in X, $\alpha\alpha^{-1} \sim \kappa_a$ in X and $\alpha^{-1}\alpha \sim \kappa_b$ in X, and
5. For a path α from a to b, and a path β from b to c, and a path γ from c to d in X, we have $(\alpha\beta)\gamma \sim \alpha(\beta\gamma)$.

Proof:

1. Exercise.
2. Exercise.
3. Let α be a path in X from a to b. We claim that $\kappa_a\alpha \sim \alpha$ in X and $\alpha\alpha^{-1} \sim \kappa_a$ in X. To show that $\kappa_a\alpha \sim \alpha$ in X, we define $F : [0,1] \times [0,1] \to X$ by

$$F(s,t) = \begin{cases} a \text{ for } 0 \leq s \leq 1, 0 \leq t \leq \frac{1-s}{2} \\ \alpha\left(\frac{2t-(1-s)}{1+s}\right) \text{ for } 0 \leq s \leq 1, \frac{1-s}{2} \leq t \leq 1 \end{cases}$$

   Verify that F is indeed a homotopy from $\kappa_a\alpha$ to α in X.
4. To show that $\alpha\alpha^{-1} \sim \kappa_a$ in X, we define a homotopy from $\alpha\alpha^{-1}$ to κ_a in X.
5. Exercise.

□

Example

The Fundamental Group of $\mathbb{S}^1$ (and of $\mathbb{C}^1$):
$\Pi_1(\mathbb{C}^*, 1) = \langle \sigma \rangle \cong \mathbb{Z}$,
$\Pi_1(\mathbb{S}^1, 1) = \langle \sigma \rangle \cong \mathbb{Z}$ where $\sigma : [0,1] \to \mathbb{S}^1 \subseteq \mathbb{C}^*$ is given by $\sigma(t) = e^{i2\pi t}$

Theorem 6.3 **Polar Coordinate Representations**

1. Path Lifting:
   Let $\alpha : [0,1] \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be a continuous map with $\alpha(0) = p$. Choose $\theta_0 \in \mathbb{R}$ so that $p = |p|e^{i\theta_0}$ (θ_0 is unique up to an integer multiple of 2π). Then there exist unique continuous maps $r, \theta : [0,1] \to \mathbb{R}$ with $r(t) > 0$ for all $t \in [0,1]$ and with $\theta(0) = \theta_0$, such that $\alpha(t) = r(t)e^{i\theta(t)}$ for all $t \in [0,1]$.
2. Homotopy Lifting:
   Let $F : [0,1] \times [0,1] \to \mathbb{C}^*$ be a continuous map with $F(0,0) = p$. Choose $\theta_0 \in \mathbb{R}$ such that $p = |p|e^{i\theta_0}$. Then there are unique continuous maps $R, \Theta : [0,1] \times [0,1] \to \mathbb{R}$ with $R(s,t) > 0$ for all s, t and with $\Theta(0,0) = \theta_0$, such that $F(s,t) = R(s,t)e^{i\Theta(s,t)}$ for all s, t.

Note

1. For α as above, if α is differentiable or C^1, or piecewise C^1, etc, then so are the maps r and θ.
2. For a continuous map $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{C} \setminus \{w\}$ with $\alpha(a) = p = |p|e^{i\theta_0}$, there exist unique continuous maps $r, \theta : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ with $r(t) > 0$ for all $t \in [a,b]$ and with $\theta(a) = \theta_0$, such that $\alpha(t) = w + r(t)e^{i\theta(t)}$ for all $t \in [a,b]$.
3. For a continuous map $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{C} \setminus \{w\}$ with $\alpha(t) = w + r(t)e^{i\theta(t)} = w + \rho(t)e^{i\varnothing(t)}$ with $r, \rho, \theta, \varnothing$ continuous, $r(t) > 0, \rho(t) > 0$ for all t, $\theta(a) = \theta_0$, $\varnothing(a) = \theta_0 + 2\pi n$, then (by uniqueness), $\rho(t) = r(t)$ for all $t \in [a,b]$ and $\varphi(t) = \theta(t) + 2\pi n$ for all $t \in [a,b]$ (since $r(t)e^{i\theta(t)} = r(t)e^{\theta(t)+2\pi n}$ for all t).

Definition 6.1.4

Let $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{C} \setminus \{w\}$ be continuous. We define the winding number $\mathrm{wind}(\alpha, w)$, of α about w as follows:
write α in polar coordinates as $\alpha(t) = w + r(t)e^{i\theta(t)}$ with $r, \theta : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ continuous with $r(t) > 0$ for all $t \in [a,b]$. Then define

$$\mathrm{wind}(\alpha, w) = \frac{\theta(b) - \theta(a)}{2\pi}$$

Note that this does not depend on the choice of $\theta(a) = \theta_0$ by Note 3.

Note

1. For α as above, if $\alpha(b) = \alpha(a)$, then $\operatorname{wind}(\alpha, w) \in \mathbb{Z}$.
2. For paths $\alpha, \beta : [0, 1] \to \mathbb{C} \setminus \{w\}$ in $\mathbb{C} \setminus \{w\}$, for $p \neq w$, $\operatorname{wind}(\kappa_p, w) = 0$ and $\operatorname{wind}(\alpha^{-1}, w) = -\operatorname{wind}(\alpha, w)$, and if $\alpha(0) = a, \alpha(1) = b = \beta(0), \beta(1) = c$, then $\operatorname{wind}(\alpha\beta, w) = \operatorname{wind}(\alpha, w) + \operatorname{wind}(\beta, w)$.
3. When $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{C} \setminus \{w\}$ is piecewise $\mathcal{C}^1$ (which means α is continuous and α' is piecewise continuous, so for $\alpha(t) = x(t) + iy(t)$, x and y are piecewise $\mathcal{C}^1$), then we can calculate $\operatorname{wind}(\alpha, w)$ using an integral as follows: For a piecewise continuous (or integrable) map $g : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ given by $g(t) = x(t) + iy(t)$, we define

$$\int_{t=a}^{b} g = \int_{t=a}^{b} g(t)dt := \int_{t=a}^{b} x(t)dt + i\int_{t=a}^{b} y(t)dt$$

When $\alpha : [a, b] \subseteq \mathbb{R} \to U \subseteq \mathbb{C}$ is piecewise $\mathcal{C}^1$ and $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is continuous, we define

$$\int_{\alpha} f = \int_{\alpha} f(z)dz := \int_{t=a}^{b} f(\alpha(t))\alpha'(t)dt$$

Given a piecewise $\mathcal{C}^1$ map $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{C} \setminus \{w\}$, writing α as $\alpha(t) = w + r(t)e^{i\theta(t)}$ with r, θ, as usual, and by letting $f(z) = \frac{1}{z-w}$ (which is continuous on $\mathbb{C} \setminus \{w\}$), we have

$$\begin{aligned}
\int_{\alpha} f &= \int_{\alpha} f(z)dz \\
&= \int_{t=a}^{b} f(\alpha(t))\alpha(t)dt \\
&= \int_{t=a}^{b} \frac{\alpha(t)'}{\alpha(t) - w}dt \\
&= \dots \\
&= \ln\left(\frac{r(b)}{r(n)}\right) + i2\pi \operatorname{wind}(\alpha, w)
\end{aligned}$$

In particular, when $\alpha(a) = \alpha(b)$, so that $r(a) = r(b)$ and $\operatorname{wind}(\alpha, w) \in \mathbb{Z}$, we have

$$\operatorname{wind}(\alpha, w) = \frac{1}{2\pi i}\int_{\alpha} \frac{dz}{z - w}$$

Theorem 6.4

Let I be an interval in $\mathbb{R}$ with $1 \in I \subseteq \mathbb{R}^+ = (0, \infty)$ and let

$$\begin{aligned} A &= \{z \in \mathbb{C} \mid \|z\| \in I\} \\ &= \left\{re^{i\theta} \in \mathbb{C} \mid r \in I, \theta \in \mathbb{R}\right\} \end{aligned}$$

Then $\pi_1(A, 1) = \langle [\delta] \rangle \cong \mathbb{Z}$, where $\delta : [0,1] \to A$ is given by $\delta(t) = e^{i2\pi t}$.

Proof: We claim that, for loops α and β at 1 in A, we have

$$\alpha \sim \beta \iff \mathrm{wind}(\alpha, 0) = \mathrm{wind}(\beta, 0)$$

Suppose $\mathrm{wind}(\alpha, 0) = \mathrm{wind}(\beta, 0) = n \in \mathbb{Z}$. Write $\alpha(t) = r(t)e^{i\theta(t)}$, $\beta(t) = \rho(t)e^{i\varnothing(t)}$, where $r, \theta, \rho, \varnothing$ are continuous, $r(t) \in I$ and $p(t) \in I$ for all $t \in [0,1]$ and $\theta(0) = \varnothing(0) = 0$. Since $\mathrm{wind}(\alpha, 0) = \mathrm{wind}(\beta, 0) = n$, we have $\theta(1) = \varnothing(1) = 2\pi n$. Then the map $F : [0,1] \times [0,1] \to A$ given by

$$F(s,t) = (r(t) + s(p(t) - r(t)))e^{i(\theta(t) + s(\varnothing(t) - \theta(t)))}$$

is a homotopy from α to β in A. The proof of the converse is in course notes. $\square$

Note

Let X be a topological space with $a \in X$, and let P be the path component of a in X. If α is a path from a to b in X, then $b \in P$ and indeed, $\alpha(t) \in P$ for all $t \in [0,1]$. If α and β are two paths from a to b in X, and F is a homotopy from α to β in X, then $F(s,t) \in P$ for all s, t (because $[0,1] \times [0,1]$ is convex, hence path-connected, and F is continuous, so $F([0,1] \times [0,1])$ is path connected with $F(0,0) = a$). It follows that $\pi_1(X, a) = \pi_1(P, a)$.

Theorem 6.5

Let γ be a path from a to b in X. Then the map $\varphi_\gamma : \pi_1(X, a) \to \pi_1(X, b)$ given by $\varphi_\gamma([\alpha]) = [\gamma\alpha\gamma^{-1}]$ is a well-defined group isomorphism (called a **change of basepoint**).

Proof: φ_γ is well-defined since for loops α and β at a in X, if $\alpha \sim \beta$ in X, then $\gamma\alpha\gamma^{-1} \cong \gamma_\beta\alpha^{-1}$. φ_γ is a group homomorphism because given two loops α and β at $a \in X$, $\gamma(\alpha\beta)\gamma^{-1} \sim \gamma\alpha\gamma^{-1}\gamma\beta\gamma^{-1}$ so that

$$\varphi_\alpha([\alpha][\beta]) = \varphi_\gamma([\alpha\beta]) = \ldots = \varphi_\gamma([\alpha]) \cdot \varphi_\gamma([\beta])$$

$\square$

Finally, we note that $\varphi_\gamma^{-1} = \varphi_{\gamma^{-1}}$.

Remark

We sometimes write $\pi_1(X, a)$ simply as $\pi_1(X)$.

Definition 6.1.5

For a topological space X, we say that X is simply connected when X is path-connected and $\pi_1(X) = 0$ (to be precise, $\pi_1(X, a) = \{[\kappa_a]\}$ for some, hence any $a \in X$).

Example

For any convex set X in a normed linear space that is simply connected (given any loop, α at a in X, the map $F : [0, 1] \times [0, 1] \to X$ given by $F(s, t) = a + s(\alpha(t) - a))$ is a homotopy from κ_a to α in X).

6.2 Invariance under homeomorphism

Definition 6.2.1

A **based topological space** is a pair (X, a) where X is a topological space and $a \in X$. We write $f : (X, a) \to (Y, b)$ to indicate that $f : X \to Y$ with $f(a) = b$.

Theorem 6.6

Given a continuous map $f : (X, a) \to (Y, b)$, the map $f_* : \pi_1(X, a) \to \pi_1(Y, b)$ given by $f_*([\alpha]) = [f \circ \alpha]$ is a well-defined group homomorphism, which we call the homomorphism of fundamental groups induced by f, or the pushforward of f.

Proof: f_* is well-defined because when α and β are loops at a in X, if $\alpha \sim \beta$ in X, then $f \circ \alpha \sim f \circ \beta$ in Y (indeed, if F is a homotopy from α to β in X, then $f \circ F$ is a homotopy from $f \circ \alpha$ to $f \circ \beta$ in Y). f_* is a group homomorphism between given loops α and β at a in X, we have $f \circ (\alpha\beta) = (f \circ \alpha)(f \circ \beta)$. Indeed,

$$(f \circ (\alpha\beta))(t) = \begin{cases} f(\alpha(2t)) \text{ if } 0 \leq t \leq \frac{1}{2} \\ f(\beta(2t-1)) \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases} = ((f \circ \alpha)(f \circ \beta))(t)$$

□

Definition 6.2.2

For a set S, we write id_S to denote the identity map on S (that is the map $\mathrm{id}_S : S \to S$) given by $\mathrm{id}_S(x) = x$ for all $x \in S$.

Theorem 6.7

1. For a based space (X, a), $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a)}$.
2. For continuous maps $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (Z, c)$, we have $(g \circ f)_* = g_* \circ f_*$.

Proof:

1. For a loop α at a in X, we have

$$(\mathrm{id}_X)_*([\alpha]) = [\mathrm{id}_X \circ \alpha] = [\alpha]$$

2. For a loop α at a in X, we have $(g \circ f) \circ \alpha = g \circ (f \circ \alpha)$ so that $(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha] = [g \circ (f \circ \alpha)] = g_*([f \circ \alpha]) = g_*(f_*[\alpha]) = (g_* \circ f_*)([\alpha])$

$\square$

Remark

Using the language of category theory, we have a covariant functor F from the category of based topological spaces (with continuous maps of based topological spaces) to the category of groups (with group homomorphisms), given by $F(X, a) = \pi_1(X, a)$ and for continuous map $f : (X, a) \to (Y, b)$, $F(f) = f_* : F(X, a) \to F(Y, b)$.

Theorem 6.8

If $f : X \to Y$ is a homeomorphism with $f(a) = b$, then $\pi_1(X, a) \cong \pi_1(Y, b)$, indeed the map $f_* : \pi_1(X, a) \to \pi_1(Y, b)$ is an isomorphism.

Proof: Let $g = f^{-1} : Y \to X$ and note that $g(b) = a$, so that we have continuous maps $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (X, a)$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ and hence $g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a)}$ and $f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y,b)}$. Thus, f_* and g_* are inverses of each other. $\square$

Example

For based spaces (X, a) and (Y, b), prove that

$$\pi_1(X \times Y, (a, b)) \cong \pi_1(X, a) \times \pi_1(Y, b)$$

6.3 Invariance under homotopy

Definition 6.3.1

1. For continuous maps $f, g : X \to Y$, a **free homotopy from f** to g is a continuous map $F : [0,1] \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all $x \in X$. When such a homotopy exists, we say that f and g are (freely) homotopic, and we write $f \sim g$.
2. For continuous maps $f, g : (X,a) \to (Y,b)$, a **homotopy from f** to g **relative to a** is a continuous map $F : [0,1] \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all $x \in X$, and $F(s,a) = f(a) = g(a) = b$ for all $s \in [0,1]$. When such a homotopy exists, we say that f and g are **homotopic relative to a**, and we write $f \sim g$ (rel a).
3. For topological spaces X and Y, and for $A \subseteq X$, and for continuous maps $f, g : X \to Y$, a **homotopy from f** to g **relative to A** is a continuous map $F : [0,1] \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all $x \in X$, and $F(s,a) = f(a) = g(a)$ for all $a \in A$ and $s \in [0,1]$. When such a homotopy exists, we say that f and g are **homotopic relative to A** and we write $f \sim g$ (rel A).

Theorem 6.9

If $f : (X,a) \to (Y,b)$ and $g : (Y,b) \to (X,a)$ are continuous with $g \circ f \sim \mathrm{id}_X$ (rel a) and $f \circ g \sim \mathrm{id}_Y$ (rel b), then $g_* \circ f_* = \mathrm{id}_{\pi_1(X,a)}$ and $f_* \circ g_* = \mathrm{id}_{\pi_1(Y,b)}$.

Theorem 6.10 **Homotopy Invariance**

If $f : X \to Y$ and $g : Y \to X$ are continuous with $f(a) = b$ and $g(b) = c$ and if $g \circ f \sim \mathrm{id}_X$ (freely) and $f \circ g$ (freely), then the maps $f_* : \pi_1(X,a) \to \pi_1(Y,b)$ and $g_* : \pi_1(Y,b) \to \pi_1(X,c)$ are isomorphisms (with $g_* \circ f_*$ equal to a change of basepoint isomorphism). When such maps f and g exist, we say that X and Y are (freely) homotopic and we write $X \sim Y$. If X and Y are path-connected with $X \sim Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Definition 6.3.2

Let X be a topological space, $A \subseteq X$ and let $i : A \to X$ be the inclusion map.

1. A **retraction map** from X to A is a continuous map $f : X \to A$ such that the restriction of f to A is the identity, or equivalently, such that $f \circ i = \mathrm{id}_A$. In this case, for any $a \in A$, $f_* \circ i_* = \mathrm{id}_{\pi_1(A,a)}$ (so f_* is surjective and i_* is injective). When such a map exists, we say that A is a **retract** of V.
2. A **strong deformation retraction map** from X to A is a continuous map $f : X \to A$ such that $f \circ i = \mathrm{id}_A$ and $i \circ f \sim \mathrm{id}_Y$ (rel A). In this case, for any $a \in A$, we also have $i \circ f \sim \mathrm{id}_X$ (rel A) and hence $f_* : \pi_1(X, a) \to \pi_1(A, a)$ and $i_* : \pi_1(A, a) \to \pi_1(X, a)$ are inverses of each other. When such a map f exists, we say that A is a **strong deformation retract** of X.

Example

$\mathbb{S}^1$ is a strong deformation retract of $\mathbb{C}^*$ given by $f(z) = \frac{z}{\|z\|}$ is a strong deformation retraction, and the map $F : [0, 1] \times C^* \to C^*$ given by

$$F(s, z) = \frac{z}{\|z\|} + s\left(z - \frac{z}{\|z\|}\right)$$

is a homotopy from $i \circ f$ to $\mathrm{id}_{\mathbb{C}^*}$ relative to $\mathbb{S}^1$.